# GROUP CLASSIFICATION OF POISSON EQUATION ON SURFACES OF REVOLUTION


BY

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*I dedicated this work to my beloved parents Late. Alh. Ma'aruf Musa Nass and Hajia Binta Aliyu, who have been giving me never-ending love and absolute support through life*

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*All praise is for Allah. We praise Him and seek His help and forgiveness. We seek refuge in Allah from evils of ourselves and the wickedness of our own deeds. Whomever Allah guides, cannot be lead astray and whoever Allah misguides, none can guide him. I bear witness that none has right to be worshipped except Allah, and I bear witness that Muhammad is his slave, and messenger. May peace of Allah be upon him.*

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# THESIS ABSTRACT


**NAME:** Aminu Ma'aruf Nass


**TITLE OF STUDY:** Group Classification of Poisson equation on Surfaces of Revolution

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Lie symmetry method is a technique to find exact solutions of differential equations. One of the important applications of Lie symmetry theory is to achieve a group classification of Lie symmetries. The group classification question for Poisson equation on manifolds $M^n (n \geq 3)$ was completely answered in [2]. However the question is open for Poisson equation on surfaces $M^2$. This thesis is concerned with carrying out a complete group of non-linear Poisson equations of the form

$$\Delta u = g(u) \tag{1}$$

on surfaces of revolution. The analysis consists of

- Finding the minimal symmetry algebra of (1).

- Determining all forms of $g(u)$ which may give larger symmetry algebras.

- Investigating the form of $g(u)$ further to classify the cases for which (1) has larger symmetry algebra, and determine the corresponding surfaces along with symmetry algebras.

# ملخص الرسالة

**الاسم:** أمينو معروف ناس

**عنوان الرسالة:** تصنيف الزمر لمعادلة بويسون على الاسطح الدروانية

**مجال التخصص:** الرياضيات

**تاريخ التخرج:** كانون الأول 2013

طريقة تناظر لي هي أحدى الطرق لإيجاد حلول دقيقة للمعادلات التفاضلية، وأحد أهم التطبيقات لنظرية تناظر لي هو الوصول للتصنيف الكامل لتناظرات لي والاختزالات التناظرية للمعادلات التفاضلية. لقد تم حل مسألة تصنيف الزمر لمعادلة بويسون على مجموعة $M^n\,(n>3)$ بشكل تام. على أي حال المسألة مفتوحة لمعادلة بويسون على سطح $M^2$ .

هذه الرسالة تعنى بعرض نتائج لتصنيف كامل لمعادلة بويسون الغير خطية على شكل

$$\triangle u = g(u) \tag{0.1}$$

على الاسطح الدروانية. التحليل يتكون من

- إيجاد الجبر التناظري الأصغري.
- أن نوجد جميع الصيغ الممكنة لدالة $g(u)$ التي تعطي جبريات تماثلية أكبر من تحديد خذه الجبريات التماثلية.
- البحث في صيغ $g(u)$ التي تعطي المعادلة $(0.1)$ جبريات تماثلية أكبر. وتحدد الاسطح المقابلة الى جانب الجبريات التماثلية.

CHAPTER 1

# PROBLEM FORMULATION

## 1.1 Introduction

The method of studying differential equations using their symmetries was introduced by Sophus Lie, who also founded the theory of infinitesimal transformations and Lie groups. Lie's classical approach is based on finding a symmetry group associated with the differential equation. This is a local Lie group of point transformations on the space of independent and dependent variables of differential equation that maps solutions to solutions. The classical method of Lie allows computing the symmetry group associated to a given differential equation. This symmetry group can further be used for many important applications in the context of differential equations. For instance, for

- determination of group invariants or similarity solutions.

- reduction of order of ODEs.

- reduction of PDEs (reduction in the number of independent variables).

- construction of new solutions from old solutions.

Hence, Lie symmetry method is a powerful method for analysing DEs and can be efficiently employed to study those problems that have an implicit or explicit symmetry. Since the modern treatment of the classical Lie symmetry theory by Ovsiannikov [11], the theory of symmetries of differential equations has been studied intensely and has substantially grown. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available, e.g. [2, 3, 4, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Let us recall that for a differential equation involving some arbitrary function $f$, the group classification problem consists of firstly finding the Lie symmetries of differential equation with arbitrary $f$ and then determining all possible forms of $f$ for which larger symmetry groups exist. The first group classification problem was carried out by Ovsiannikov [11] who classified all forms of the non-linear heat equation $u_t = (f(u)u_x)_x$. Since then a number of articles on symmetry analysis and classification problem have appeared in literature, cf. [6, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. This work deals with group classification problem of Poisson equation (1.6) with the arbitrary function introduced through the general metric (1.5). The calculations of minimal symmetry algebra and the forms of $g$ which provide larger symmetry algebra are based on necessary conditions on $g$ obtained through a triangulation of determining equations of Lie symmetries. An efficient method to obtain triangulation is the well-known method of Mansfield [31] of generating differential Grobner bases of determining equations using Kolchin-Ritt algorithm. In this thesis we intend to obtain a triangulation of the determining equations using a variant of Kolchin-Ritt algorithm, the direct search algorithm by Clarkson-Mansfield [25], that allows to achieve a faster triangulation of determining equations. We refer to [25] for a detailed discussion of these algorithms. This method was also implemented in [31] to investigate the group classification problem for a class of Klein Gordon equations.

The symmetry properties and reductions of most of the fundamental equations of mathematical physics, with flat background metric, have been well investigated; cf. [3, 4, 5]. In case of non-flat background metric, some studies of heat or wave equation, using symmetries, on specific cases of surfaces like sphere, torus, cone and hyperbolic space have also been carried out in a series of recent papers like [6, 7, 8, 9, 10].

Physically significant and geometrically rich classes of surfaces include surfaces of revolution, ruled surfaces, tubular surfaces, and minimal surfaces. These classes of surfaces possess a wide range of applications, for example, in computer graphics, digital design, architecture, engineering design, study of biological membranes, sheet metal based industries, the study of key objects in most nonlinear phenomena in physics and field theories etc. Surfaces of revolution form a large class of surfaces, which are generated by rotating a plane curve about an axis. Hence, due to their construction, such surfaces naturally possess nice symmetry properties. This makes surfaces of revolutions and related problems an interesting area of contemporary research, and particularly of importance in the fields of physics, engineering, computer graphics and other disciplines involving models of physical processes with natural symmetries. Well known examples of surfaces of revolution include cylinder, cone, sphere, hyperboloid, ellipsoid, Gabriel's horn, pseudosphere, torus, catenoid and tractoid. The motivation for the study of this thesis is twofold. A classification of surfaces of revolution according to their isometries was carried out by Eisenhart in 1925 [1]. It was proved that there are only two classes of surfaces of revolution according to their isometries. The minimal isometry algebra of a surface of revolution is 1-dimensional and the larger isometry algebra, which is 3-dimensional, is admitted only for surfaces with constant curvature. Eisenhart's classification provides the first motivation for this work where we investigate group classification of non-linear Poisson equation on surfaces of revolution. An interesting question from the outset is whether the class of surfaces for which the minimal symmetry algebra extends includes any surfaces of non-constant curvature.

The second motivation to investigate the group classification problem for the Poisson equation on general surfaces of revolution is as follows. The group classification question for Poisson equation on higher dimensional manifolds $M^n$ ($n \geq 3$) has been completely answered in [2]. However, the question is open for Poisson equation on surfaces. In this thesis we consider non-linear Poisson equation on a general surface of revolution and carry out a complete group classification.

## 1.2 Basic definitions from differential geometry

For us to formulate our problem, we need to give some definitions from differential geometry [36]. Surface of revolution obtained by rotating a plane curve about an axis a form a large class of surfaces. Therefore, surfaces of revolution naturally posses nice symmetry properties.

This makes them and related problems an interesting area of contemporary research and particularly of importance in the fields of applied mathematics; computer graphics and other disciplines involve modeling of physical processes with natural symmetries. Well known examples of surfaces of revolution include cylinder, cone, sphere, hyperboloid, ellipsoid, Gabriel's horn, pseudosphere, torus, catenoid and tractoid.

**Definition 1.1** *A parametrized surface $X : D \to \mathbb{R}^3$ is a smooth function of an open set $D \subset \mathbb{R}^2$ into $\mathbb{R}^3$, defined by*

$$X(u, v) = (X_1(u, v), X_2(u, v), X_3(u, v)).$$

**Definition 1.2** *A regular parametrization is a function $X : D \to \mathbb{R}^3$ of an open set $D \subset \mathbb{R}^2$, defined by*

$$X(u, v) = (X_1(u, v), X_2(u, v), X_3(u, v)),$$

*satisfying the following conditions:*

- *$X$ is differentiable i.e. the functions $X_i$ have all partial derivatives of all order.*
- *$X$ is one-to-one.*
- *$X_u$ and $X_v$, are linearly independent.*

**Definition 1.3** *Let $X(x, y) : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrization of surface then, the Riemannian metric or first fundamental form of the patch $X$ is defined by*

$$g = ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

*with coefficient of the first fundamental form defined by*

$$E = X_x \cdot X_x, \qquad F = X_x \cdot X_y, \qquad G = X_y \cdot X_y. \tag{1.1}$$

**Definition 1.4** *Matrix of First Fundamental form:*

*Let*

$$g_{11} = E = X_x \cdot X_x, \qquad g_{12} = F = X_x \cdot X_y, \qquad g_{22} = G = X_y \cdot X_y. \tag{1.2}$$

*Then it is often convenient to put the metric as*

$$g = ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$$

*where the symmetric matrix form is defined by*

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{ij}.$$

*The inverse of $g$ is*

$$g^{-1} = \frac{1}{det(g)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} = g^{ij},$$

*with*

$$det(g) = g_{11}g_{22} - g_{12}^2.$$

**Definition 1.5** *For any metric $g$, the Laplacian on $(S, g)$ is defined by*

$$\Delta u = \frac{1}{\sqrt{| g |}} \frac{\partial}{\partial x^i} (\sqrt{| g |} g^{ij} \frac{\partial}{\partial x^j} (u))$$

**Definition 1.6** *Let $X(x, y)$ be a regular parametrization of a surface, with*

$$E = X_x \cdot X_x, \qquad F = X_x \cdot X_y, \qquad G = X_y \cdot X_y, \qquad N = \frac{X_x \times X_y}{|X_x \times X_y|}.$$

*The second fundamental form of patch $X(x, y)$ is defined as*

$$ds^2 = l dx^2 + m dx dy + n dy^2,$$

*with coefficients of the second fundamental form defined by*

$$l = X_{xx} \cdot N, \qquad m = X_{xy} \cdot N, \qquad n = X_{yy} \cdot N, \qquad N = \frac{X_x \times X_y}{|X_x \times X_y|}.$$

*The Gaussian curvature is the product of the maximum and minimum values of the second fundamental form restricted to the unit tangent vectors. The Gaussian curvature of surface is given by the formula* [39]

$$K = \frac{ln - m^2}{EG - F^2}.$$

### 1.2.1 Gaussian curvature of surfaces of revolution

Since in this thesis we consider different surfaces of revolution with different Gaussian curvature, we give the detail derivation of the formula of Gaussian curvature on surfaces of revolution.

**Theorem 1.1** *The Gaussian curvature of surface of revolution generated by a unit speed curve $\alpha(x) = (v(x), w(x))$ with regular parametrizations*

$$X(x, y) = (v(x), w(x)\cos(y), w(x)\sin(y))$$

*is given by the formula*

$$K = -\frac{w''(x)}{w(x)}$$

**Proof.** From the formula of the coefficients of first fundamental form

$$E = X_x \cdot X_x = (v'(x), w'(x)\cos(y), w'(x)\sin(y)) \cdot (v'(x), w'(x)\cos(y), w'(x)\sin(y)).$$

Since $\alpha(x)$ is a unit speed curve, this implies $E = v'^2(x) + w'^2(x) = 1$ .

$$F = X_x \cdot X_y = (v'(x), w'(x)\cos(y), w'(x)\sin(y)) \cdot (0, -w(x)\sin(y), w(x)\cos(y)) = 0,$$

$$G = X_y \cdot X_y = (0, -w(x)\sin(y), w(x)\cos(y)) \cdot (0, -w(x)\sin(y), w(x)\cos(y)) = w^2(x).$$

So from the formula of the coefficients second fundamental form

$$N = (w'(x), -v'(x)\cos(y), -v'(x)\sin(y)),$$

$$l = (v''(x), w''(x)\cos(y), w''(x)\sin(y)) \cdot (w'(x), -v'(x)\cos(y), -v'(x)\sin(y)).$$

Therefore

$$l = v''(x)w'(x) - v'(x)w''(x),$$

$$m = (0, -w'(x)\sin(y), w'(x)\cos(y)) \cdot (w'(x), -v'(x)\cos(y), -v'(x)\sin(y)) = 0,$$

$$n = (0, -w(x)\cos(y), -w(x)\sin(y)) \cdot (w'(x), -v'(x)\cos(y), -v'(x)\sin(y)) = v'(x)w(x).$$

Finally the Gaussian curvature is

$$K = \frac{v'(x)(w'(x)v''(x) - w''(x)v'(x))}{w(x)}. \tag{1.3}$$

Since $\alpha(x)$ is a unit speed curve, we have

$$v'(x)v''(x) + w'(x)w''(x) = 0. \tag{1.4}$$

Putting (1.4) in (1.3), we get

$$K = -\frac{w''(x)}{w(x)}$$

. ∎

**Definition 1.7** *A surface of revolution whose Gaussian curvature is identically zero is called flat surfaces of revolution.*

### 1.2.2 Problem formulation

Consider the Patch of surface of revolution

$$X(x, y) = (v(x), w(x)\cos(y), w(x)\sin(y)),$$

obtained by rotating a unit speed cure $\alpha(x) = (v(x), w(x))$, with first fundamental for

$$g = dx^2 + w^2(x)dy^2.$$

Therefore Laplacian on surface of revolution is given by

$$\Delta u = \frac{w'(x)}{w(x)}u_x + u_{xx} + \frac{1}{w^2(x)}u_{yy} = 0.$$

To ensure regularity of $X$, since $|X_x \times X_y|^2 = w^2(x)$, we assume $w(x) > 0$, that is, the curve $\alpha(x)$ does not intersect the axis of rotation. Let assume $w(x) = e^{f(x)}$ for some smooth function $f(x)$. Therefore the metric becomes

$$g = dx^2 + e^{2f(x)}dy^2. \tag{1.5}$$

It then follow directly that the Poisson equation on the surfaces of revolution takes the form

$$\Delta u = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} = g(u). \tag{1.6}$$

# CHAPTER 2

# SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we focus on basic definitions and results in Lie symmetry method for partial differential equations(PDEs) that will provide the necessary background for the research work carried out in this thesis.The short review consists of presentation of the notion of Lie symmetries of PDEs, the discussion of prolongations and the procedure for finding symmetries of PDEs.

Throughout this chapter, we shall restrict our work to second order PDEs with one dependent variable and two independent variables. This will not in any case effect our results in the chapters 3 of our work since the PDEs involved all belong to this class of PDEs. Idea about the subject of Lie symmetry for general PDEs is contained in many standard books in literature cf [[4],[14],[16],[19],[21]].

Consider a 2nd order partial differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2.1}$$

A one parameter group of transformations $x^* = f(x, y, u, \varepsilon)$,$y^* = g(x, y, u, \varepsilon)$, $u^* = h(x, y, u, \varepsilon)$ is a symmetry of PDE (2.1) if the PDE (2.1) is invariant under the transformation $(x, y, u) \to (x^*, y^*, u^*)$, i.e. after change of variables $(x, y, u) \to (x^*, y^*, u^*)$ we have

$$F(x^*, y^*, u^*, u^*_x, u^*_y, u^*_{xx}, u^*_{xy}, u^*_{yy}) = 0. \tag{2.2}$$

We shall represent the transformation via their Taylor series expansion with respect to the parameter in the neighbourhood of $\epsilon = 0$ and write the infinitesimal form of these transformation as follows.

$$x^* = x + \epsilon\xi(x, y, u) + O(\epsilon^2), \tag{2.3}$$

$$y^* = y + \epsilon\tau(x, y, u) + O(\epsilon^2), \tag{2.4}$$

$$u^* = u + \epsilon\phi(x, y, u) + O(\varepsilon^2), \tag{2.5}$$

with $\xi$ , $\tau$ and $\phi$ given by

$$\xi(x,y,u) = \frac{\partial}{\partial \epsilon} f|_{\epsilon=0}, \qquad \tau(x,y,u) = \frac{\partial}{\partial \epsilon} g|_{\epsilon=0}, \qquad \phi(x,y,u) = \frac{\partial}{\partial \epsilon} h|_{\epsilon=0}. \tag{2.6}$$

Hence the tangent vector field to the curve $\alpha(\epsilon) = (x^*, y^*, u^*)$ is

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}, \tag{2.7}$$

and is called the infinitesimal generator or operator of the one parameter group.

## 2.1 Prolongation of infinitesimal generators of symmetries of PDEs

The symmetry operator (2.7) provides the information on how the variables $x$,$y$ and $u$ are transformed. However this information is not enough as the symmetry analysis of PDEs also require information about transformation of derivative. In this section we discuss how to prolong infinitesimal generators to obtain the complete information of variables of the PDE (2.1) as well as the derivatives. The discussion of prolongation formulas is restricted to symmetries of 2nd order PDEs of the form

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0$$

, with symmetry operator

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

or equivalently the infinitesimal transformations

$$x^* = x + \epsilon\xi(x,y,u) + O(\epsilon^2), \qquad y^* = y + \epsilon\tau(x,y,u) + O(\epsilon^2), \qquad u^* = u + \epsilon\phi(x,y,u) + O(\varepsilon^2).$$

It is convenient to use the operator of total differentiation for writing the prolongation.

**Definition 2.1** *Consider the function*

$$F(x, y, u(x,y), g_1(x,y), g_2(x,y), \ldots, g_n(x,y)) = 0.$$

*The total differentiation operators with respect to $x$ and $y$ are defined respectively as*

$$D_x = \frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial g_1}{\partial x}\frac{\partial}{\partial g_1} + \cdots + \frac{\partial g_n}{\partial x}\frac{\partial}{\partial g_n},$$

$$D_y = \frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial g_1}{\partial y}\frac{\partial}{\partial g_1} + \cdots + \frac{\partial g_n}{\partial y}\frac{\partial}{\partial g_n}.$$

As an example, consider the PDEs (2.1), the total differentiation operators with respect to $x$ and $y$ take the form

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y},$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y}.$$

### 2.1.1 First prolongation of infinitesimal generator X

Given a PDE and a one parameter group of infinitesimal transformations

$$x^* = x + \epsilon\xi(x, y) + O(\epsilon),$$

$$y^* = y + \epsilon\tau(x, y) + O(\epsilon),$$

$$u^* = u + \epsilon\phi(x, y) + O(\epsilon),$$

with the corresponding infinitesimal generator of the form,

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}.$$

We want to find the transformation of the first order partial derivatives of $u$ with respect to $x$ and $y$ i.e. we need to obtain the functions $\eta^{[x]}(x, y, u, u_x, u_y)$ and $\eta^{[y]}(x, y, u, u_x, u_y)$ such that

$$u^*_{x^*} = u_x + \epsilon\eta^{[x]}(x, y, u, u_x, u_y) + O(\epsilon^2), \tag{2.8}$$

$$u^*_{y^*} = u_y + \epsilon\eta^{[y]}(x, y, u, u_x, u_y) + O(\epsilon^2). \tag{2.9}$$

From (2.3), we have

$$
\begin{aligned}
dx^* &= dx + \varepsilon d\xi + O(\varepsilon^2) \\
&= dx + \epsilon(\xi_x dx + \xi_y dy + \xi_u du) + O(\varepsilon^2) \\
&= dx + \epsilon(\xi_x dx + \xi_y dy + \xi_u(\tfrac{\partial u}{\partial x} dx + \tfrac{\partial u}{\partial y} dy)) + O(\epsilon^2) \\
&= (1 + \epsilon(\xi_x + \xi_u \tfrac{\partial u}{\partial x}))dx + \epsilon(\xi_y + \xi_u \tfrac{\partial u}{\partial y})dy + O(\epsilon^2),
\end{aligned}
$$

which gives

$$dx^* = (1 + \varepsilon D_x \xi)dx + \epsilon(D_y \xi)dy + O(\epsilon^2). \tag{2.10}$$

Following same pattern and using (2.4), we have

$$dy^* = \epsilon(D_x \tau)dx + (1 + \epsilon D_y \tau)dy + O(\epsilon^2). \tag{2.11}$$

From equation (2.5), we have

$$
\begin{aligned}
du^* &= du + \epsilon d\phi + O(\epsilon^2) \\
&= \tfrac{\partial u}{\partial x} dx + \tfrac{\partial u}{\partial y} dy + \epsilon(\phi_x dx + \phi_y dy + \phi_u(\tfrac{\partial u}{\partial x} dx + \tfrac{\partial u}{\partial y} dy +)) + O(\epsilon^2) \\
&= (\tfrac{\partial u}{\partial x} + \varepsilon(\phi_x + \phi_u \tfrac{\partial u}{\partial x}))dx + (\tfrac{\partial u}{\partial y} + \epsilon(\phi_y + \phi_u \tfrac{\partial u}{\partial y}))dy + O(\epsilon^2),
\end{aligned}
$$

which gives

$$du^* = (\frac{\partial u}{\partial x} + \epsilon D_x \phi)dx + (\frac{\partial u}{\partial y} + \epsilon D_y \phi)dy + O(\epsilon^2). \tag{2.12}$$

Also $u^* = u^*(x^*, y^*)$, implies that

$$du^* = \frac{\partial u^*}{\partial x^*}dx^* + \frac{\partial u^*}{\partial y^*}dy^*. \tag{2.13}$$

Using equations (2.10) and (2.11) in (2.13), and re-arranging gives

$$\begin{aligned}&(\tfrac{\partial u}{\partial x} + \epsilon D_x\phi)dx + (\tfrac{\partial u}{\partial y} + \epsilon D_y\phi)dy + O(\epsilon^2)\\ &= \left\{\tfrac{\partial u^*}{\partial x^*}(1 + \epsilon D_x\xi) + \epsilon\tfrac{\partial u^*}{\partial y^*}(D_x\tau)\right\} dx + \left\{\epsilon\tfrac{\partial u^*}{\partial x^*}(D_y\xi) + \varepsilon\tfrac{\partial u^*}{\partial t^*}(D_y\tau) + \tfrac{\partial u^*}{\partial y^*}(1 + \epsilon D_y\tau)\right\} dy + O(\epsilon^2).\end{aligned}$$

Since $dx$ and $dy$ are linearly independent, the above relation implies that

$$\frac{\partial u}{\partial x} + \epsilon D_x\phi = \frac{\partial u^*}{\partial x^*}(1 + \epsilon D_x\xi) + \epsilon\frac{\partial u^*}{\partial y^*}(D_x\tau), \tag{2.14}$$

$$\frac{\partial u}{\partial y} + \epsilon D_y\phi = \varepsilon\frac{\partial u^*}{\partial x^*}(D_y\xi) + \frac{\partial u^*}{\partial y^*}(1 + \epsilon D_y\tau). \tag{2.15}$$

Next we express the system (2.14) and (2.15) as a matrix below

$$\begin{pmatrix} \frac{\partial u}{\partial x} + \epsilon D_x\phi \\ \frac{\partial u}{\partial y} + \epsilon D_y\phi \end{pmatrix} = \begin{pmatrix} 1 + \epsilon D_x\xi & \epsilon D_x\tau \\ \epsilon D_y\xi & 1 + \epsilon D_y\tau \end{pmatrix} \begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} \end{pmatrix}. \tag{2.16}$$

Let us define $A$ and $B$ by

$$B = \begin{pmatrix} D_x\xi & D_x\tau & D_x\tau \\ \\ D_y\xi & D_y\tau & D_y\tau \end{pmatrix}, \tag{2.17}$$

$$A = \begin{pmatrix} 1+\epsilon D_x\xi & \epsilon D_x\tau \\ \\ \epsilon D_y\xi & 1+\epsilon D_y\tau \end{pmatrix}, \tag{2.18}$$

$$A^{-1} = (I+\varepsilon B)^{-1} = I - \epsilon B + O(\epsilon^2). \tag{2.19}$$

Then, from equation (2.16), we get

$$\begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \\ \frac{\partial u^*}{\partial y^*} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial u}{\partial x} + \varepsilon D_x\phi \\ \\ \frac{\partial u}{\partial y} + \varepsilon D_y\phi \end{pmatrix} + O(\epsilon^2). \tag{2.20}$$

This is same as

$$\begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \\ \frac{\partial u^*}{\partial y^*} \end{pmatrix} = (I - \epsilon B) \begin{pmatrix} \frac{\partial u}{\partial x} + \varepsilon D_x\phi \\ \\ \frac{\partial u}{\partial y} + \varepsilon D_y\phi \end{pmatrix} + O(\varepsilon^2). \tag{2.21}$$

Using (2.17) and equation (2.21), we have

$$\begin{pmatrix} \frac{\partial u}{\partial x} + \epsilon \eta^{[x]}(x, y, u, u_x, u_y) + O(\epsilon^2) \\ \\ \frac{\partial u}{\partial y} + \epsilon \eta^{[y]}(x, y, u, u_x, u_y) + O(\epsilon^2) \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \\ \frac{\partial u}{\partial y} \end{pmatrix} + \epsilon \begin{pmatrix} D_x \phi \\ \\ D_y \phi \end{pmatrix} - \varepsilon B \begin{pmatrix} \frac{\partial u}{\partial x} \\ \\ \frac{\partial u}{\partial y} \end{pmatrix} + O(\epsilon^2). \tag{2.22}$$

This gives

$$\begin{pmatrix} \eta^{[x]} \\ \\ \eta^{[y]} \end{pmatrix} = \begin{pmatrix} D_x \varphi \\ \\ D_y \varphi \end{pmatrix} - \begin{pmatrix} D_x \xi & D_x \phi \\ \\ D_y \xi & D_y \tau \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \\ \frac{\partial u}{\partial y} \end{pmatrix}. \tag{2.23}$$

Finally, from (2.23) we can write $\eta^{[x]}$ and $\eta^{[y]}$ with respect to $\xi$, $\tau$ and $\phi$ as

$$\eta^{[x]} = D_x \phi - (u_x D_x \xi + u_y D_x \tau). \tag{2.24}$$

Therefore

$$\eta^{[x]} = \phi_x + (\phi_u - \xi_x) u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y. \tag{2.25}$$

In the same way

$$\eta^{[y]} = \phi_y - \xi_y u_x + (\phi_u - \tau_y) u_y - \xi_u u_x u_y - \tau_u u_y^2. \tag{2.26}$$

Therefore we can write the first prolongation as

$$X^{[1]} = X + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y},$$

with $\eta^{[x]}$ and $\eta^{[y]}$ given by (2.25) and (2.26) respectively.

### 2.1.2 Second prolongation of infinitesimal generator X

Consider the first prolongation of the operator X

$$X^{[1]} = X + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y}.$$

Next we look for the prolongation of the all second order partial derivative of $u$ with respect to $x$ and $y$ that is $\eta^{[xx]}, \eta^{[xy]}, \eta^{[yy]}$ such that

$$u^*_{x^*x^*} = u_{xx} + \epsilon\eta^{[xx]}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) + O(\epsilon^2), \tag{2.27}$$

$$u^*_{x^*y^*} = u_{xy} + \epsilon\eta^{[xy]}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) + O(\epsilon^2), \tag{2.28}$$

$$u^*_{y^*y^*} = u_{yy} + \varepsilon\eta^{[yy]}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) + O(\epsilon^2). \tag{2.29}$$

From (2.8), we have

$$du^*_{x^*} = du_x + \epsilon d\eta^{[x]} + O(\epsilon^2) \tag{2.30}$$

$$= \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \epsilon(\frac{\partial \eta^{[x]}}{\partial x}dx + \frac{\partial \eta^{[x]}}{\partial y}dy + \frac{\partial \eta^{[x]}}{\partial u}du + \frac{\partial \eta^{[x]}}{\partial u_x}du_x + \frac{\partial \eta^{[x]}}{\partial u_y}du_y) + O(\varepsilon^2)$$

$$= (\frac{\partial u_x}{\partial x} + \epsilon(\frac{\partial \eta^{[x]}}{\partial x} + \frac{\partial \eta^{[x]}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \eta^{[x]}}{\partial u_x}\frac{\partial u_x}{\partial x} + \frac{\partial \eta^{[x]}}{\partial u_y}\frac{\partial u_y}{\partial x}))dx + (\frac{\partial u_x}{\partial y} + \epsilon(\frac{\partial \eta^{[x]}}{\partial y} + \frac{\partial \eta^{[x]}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \eta^{[x]}}{\partial u_x}\frac{\partial u_x}{\partial y} + \frac{\partial \eta^{[x]}}{\partial u_y}\frac{\partial u_y}{\partial y}))dy + O(\epsilon^2),$$

which gives

$$du^*_{x^*} = (\frac{\partial u_x}{\partial x} + \epsilon D_x \eta^{[x]})dx + (\frac{\partial u_x}{\partial y} + \epsilon D_y \eta^{[x]})dy + O(\epsilon^2).$$

Putting equations (2.10) and (2.11) in

$$du^*_{x^*} = \frac{\partial u^*_{x^*}}{\partial x^*}dx^* + \frac{\partial u^*_{x^*}}{\partial y^*}dy^*,$$

we have

$$du^*_{x^*} = u^*_{x^*x^*}([1 + \epsilon D_x\xi]dx + \epsilon[D_y\xi]dy) + u^*_{x^*y^*}(\epsilon[D_x\tau]dx + [1 + \varepsilon D_y\tau]dy) + O(\epsilon^2).$$

Using (2.25) and the fact that $dx$ and $dy$ are independent, we have

$$\frac{\partial u_x}{\partial x} + \varepsilon D_x\eta^{[x]} = u^*_{x^*x^*}(1 + \epsilon D_x\xi) + \epsilon u^*_{x^*y^*}(D_x\tau)$$

$$\frac{\partial u_x}{\partial y} + \varepsilon D_y\eta^{[x]} = \epsilon u^*_{x^*x^*}(D_y\xi) + u^*_{x^*y^*}(1 + \epsilon D_y\tau).$$

The matrix form of this is given by

$$\begin{pmatrix} u_{xx} + \epsilon D_x \eta^{[x]} \\ \\ u_{xy} + \epsilon D_y \eta^{[x]} \end{pmatrix} = \begin{pmatrix} 1 + \epsilon D_x \xi & \epsilon D_x \tau \\ \\ \epsilon D_y \xi & 1 + \epsilon D_y \tau \end{pmatrix} \begin{pmatrix} u^*_{x^*x^*} \\ \\ u^*_{x^*y^*} \end{pmatrix} + O(\epsilon^2).$$

This implies

$$\begin{pmatrix} u_{xx} + \epsilon D_x \eta^{[x]} \\ \\ u_{xy} + \epsilon D_y \eta^{[x]} \end{pmatrix} = A \begin{pmatrix} u^*_{x^*x^*} \\ \\ u^*_{x^*y^*} \end{pmatrix} + O(\epsilon^2),$$

or equivalently as

$$\begin{pmatrix} u_{xx} + \epsilon D_x \eta^{[x]} \\ \\ u_{xy} + \epsilon D_y \eta^{[x]} \end{pmatrix} = (I + \epsilon B) \begin{pmatrix} u^*_{x^*x^*} \\ \\ u^*_{x^*y^*} \end{pmatrix} + O(\epsilon^2)$$

with $B$ and $A$ as defined in the previous sub-section. Therefore

$$\begin{pmatrix} u^*_{x^*x^*} \\ \\ u^*_{x^*y^*} \end{pmatrix} = (I + \varepsilon B)^{-1} \begin{pmatrix} u_{xx} + \epsilon D_x \eta^{[x]} \\ \\ u_{xy} + \epsilon D_y \eta^{[x]} \end{pmatrix} + O(\epsilon^2),$$

$$\begin{pmatrix} u^*_{x^*x^*} \\ \\ u^*_{x^*y^*} \end{pmatrix} = (I - \epsilon B) \begin{pmatrix} u_{xx} + \epsilon D_x \eta^{[x]} \\ \\ u_{xy} + \epsilon D_y \eta^{[x]} \end{pmatrix} + O(\epsilon^2)$$

which implies

$$\begin{pmatrix} u^*_{x^*x^*} \\ \\ u^*_{x^*y^*} \end{pmatrix} = \begin{pmatrix} u_{xx} \\ \\ u_{xy} \end{pmatrix} + \varepsilon \left( \begin{pmatrix} D_x \eta^{[x]} \\ \\ D_y \eta^{[x]} \end{pmatrix} - B \begin{pmatrix} u_{xx} \\ \\ u_{xy} \end{pmatrix} \right) + O(\epsilon^2).$$

Finally, using [(2.27),(2.28) and (2.29)], we have

$$\begin{pmatrix} \eta^{[xx]} \\ \\ \eta^{[xy]} \end{pmatrix} = \begin{pmatrix} D_x\eta^{[x]} \\ \\ D_y\eta^{[x]} \end{pmatrix} - B \begin{pmatrix} u_{xx} \\ \\ u_{xy} \end{pmatrix} \tag{2.31}$$

with

$$B = \begin{pmatrix} D_x\xi & D_x\tau \\ \\ D_y\xi & D_y\tau \end{pmatrix}, \tag{2.32}$$

$$\eta^{[xx]} = D_x\eta^{[x]} - u_{xx}D_x\xi - u_{xy}D_x\tau. \tag{2.33}$$

Therefore

$$\eta^{[xx]} = D_x\eta^{[x]} - u_{xx}(\xi_x + u_x\xi_u) - u_{xy}(\tau_x + u_x\tau_u). \tag{2.34}$$

From (2.25), we have

$$\eta^{[x]} = \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y.$$

Therefore

$$D_x\eta^{[x]} = (\tfrac{\partial}{\partial x} + u_x\tfrac{\partial}{\partial u} + u_{xx}\tfrac{\partial}{\partial u_x} + u_{xy}\tfrac{\partial}{\partial u_y})\eta^{[x]}. \tag{2.35}$$

Substituting $\eta^{[x]}$ in (2.35), we have

$D_x\eta^{[x]} = \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_xu_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2u_y + (\phi_u - \xi_x)u_{xx} - \tau_xu_{xy} - 2\xi_uu_{xx}u_x - \tau_uu_{xx}u_y - \tau_uu_{xy}u_x.$

Finally

$\eta^{[xx]} = \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_xu_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2u_y + (\phi_u - 2\xi_x)u_{xx} - 2\tau_xu_{xy} - 3\xi_uu_{xx}u_x - \tau_uu_{xx}u_y - 2\tau_uu_{xy}u_x.$

Similarly we can find $\eta^{[xy]}$ as follow:

$$D_y\eta^{[x]} = (\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y})\eta^{[x]} \tag{2.36}$$

Substituting $\eta^{[x]}$ in (2.36), we have

$D_y\eta^{[x]} = \phi_{xy} + (2\phi_{uy} - \xi_{xy})u_x - \tau_{xyy}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_xu_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2u_y + (\phi_u - \xi_x)u_{xx} - \tau_xu_{xy} - 2\xi_uu_{xx}u_x - \tau_uu_{xx}u_y - \tau_uu_{xy}u_x.$

Finally

$\eta^{[xy]} = \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_xu_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2u_y + (\phi_u - 2\xi_x)u_{xx} - 2\tau_xu_{xy} - 3\xi_uu_{xx}u_x - \tau_uu_{xx}u_y - 2\tau_uu_{xy}u_x.$

To obtain $\eta^{[yy]}$ we follow same procedure.

$$du^*_{y^*} = du_y + \epsilon d\eta^{[y]} + O(\epsilon^2), \tag{2.37}$$

$$du^*_{y^*} = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \epsilon(\frac{\partial\eta^{[y]}}{\partial x}dx + \frac{\partial\eta^{[y]}}{\partial y}dy + \frac{\partial\eta^{[y]}}{\partial u}du + \frac{\partial\eta^{[y]}}{\partial u_x}du_x + \frac{\partial\eta^{[y]}}{\partial u_y}du_y) + O(\epsilon^2). \tag{2.38}$$

Therefore

$$du^*_{y^*} = (\tfrac{\partial u_y}{\partial x} + \epsilon(\tfrac{\partial \eta^{[y]}}{\partial x} + \tfrac{\partial \eta^{[y]}}{\partial u}\tfrac{\partial u}{\partial x} + \tfrac{\partial \eta^{[y]}}{\partial u_x}\tfrac{\partial u_x}{\partial x} + \tfrac{\partial \eta^{[y]}}{\partial u_y}\tfrac{\partial u_y}{\partial x}))dx + (\tfrac{\partial u_y}{\partial y} + \epsilon(\tfrac{\partial \eta^{[y]}}{\partial y} + \tfrac{\partial \eta^{[y]}}{\partial u}\tfrac{\partial u}{\partial y} + \tfrac{\partial \eta^{[y]}}{\partial u_x}\tfrac{\partial u_x}{\partial y} + \tfrac{\partial \eta^{[y]}}{\partial u_y}\tfrac{\partial u_y}{\partial y}))dy + O(\epsilon^2) \tag{2.39}$$

which implies

$$du^*_{y^*} = (\frac{\partial u_y}{\partial x} + \epsilon D_x \eta^{[y]})dx + (\frac{\partial u_y}{\partial y} + \epsilon D_y \eta^{[y]})dy + O(\varepsilon^2). \tag{2.40}$$

Using equations (2.10) and (2.11) in

$$du^*_{y^*} = \frac{\partial u^*_{y^*}}{\partial x^*}dx^* + \frac{\partial u^*_{y^*}}{\partial y^*}dy^*, \tag{2.41}$$

gives

$$du^*_{y^*} = u^*_{x^*y^*}((1 + \epsilon D_x \xi)dx + \epsilon(D_y \xi)dy) + u^*_{y^*y^*}(\epsilon(D_x \tau)dx + (1 + \epsilon D_y \tau)dy) + O(\epsilon^2). \tag{2.42}$$

Using (2.25) and the fact that $dx$ and $dy$ are independent, we have

$$\tfrac{\partial u_y}{\partial x} + \varepsilon D_x \eta^{[y]} = u^*_{x^*y^*}(1 + \epsilon D_x \xi) + \epsilon u^*_{y^*y^*}(D_x \tau), \tag{2.43}$$

$$\tfrac{\partial u_y}{\partial y} + \varepsilon D_y \eta^{[y]} = \epsilon u^*_{x^*y^*}(D_y \xi) + u^*_{y^*y^*}(1 + \epsilon D_y \tau). \tag{2.44}$$

Next we express the above in a matrix form

$$\begin{pmatrix} u_{xy}+\varepsilon D_x\eta^{[y]} \\ u_{yy}+\varepsilon D_y\eta^{[y]} \end{pmatrix} = \begin{pmatrix} 1+\varepsilon D_x\xi & \varepsilon D_x\tau \\ \varepsilon D_y\xi & 1+\varepsilon D_y\tau \end{pmatrix} \begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \end{pmatrix} + O(\varepsilon^2),$$

$$\begin{pmatrix} u_{xy}+\epsilon D_x\eta^{[y]} \\ u_{yy}+\epsilon D_y\eta^{[y]} \end{pmatrix} = A \begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \end{pmatrix} + O(\epsilon^2) = (I+\epsilon B) \begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \end{pmatrix} + O(\epsilon^2),$$

$$\begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \end{pmatrix} = (I+\epsilon B)^{-1} \begin{pmatrix} u_{xy}+\epsilon D_x\eta^{[y]} \\ u_{yy}+\epsilon D_y\eta^{[y]} \end{pmatrix} + O(\epsilon^2),$$

$$\begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \end{pmatrix} = (I-\epsilon B) \begin{pmatrix} u_{xy}+\epsilon D_x\eta^{[y]} \\ u_{yy}+\epsilon D_y\eta^{[y]} \end{pmatrix} + O(\epsilon^2),$$

$$\begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \end{pmatrix} = \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix} + \varepsilon \left( \begin{pmatrix} D_x\eta^{[y]} \\ D_y\eta^{[y]} \end{pmatrix} - B \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix} \right) + O(\varepsilon^2).$$

Using (2.27), (2.28) and (2.29), we have

$$\begin{pmatrix} \eta^{[xy]} \\ \eta^{[yy]} \end{pmatrix} = \begin{pmatrix} D_x\eta^{[y]} \\ D_y\eta^{[y]} \end{pmatrix} - B \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix}, \tag{2.45}$$

with $B$ given by

$$B = \begin{pmatrix} D_x\xi & D_x\tau \\ D_y\xi & D_y\tau \end{pmatrix}.$$

This implies that

$$\eta^{[yy]} = D_y\eta^{[y]} - u_{xy}D_y\xi - u_{yy}D_y\tau, \tag{2.46}$$

$$\eta^{[yy]} = D_y\eta^{[y]} - u_{xy}(\xi_y + u_y\xi_u) - u_{yy}(\tau_y + u_y\tau_u). \tag{2.47}$$

But from equation (2.21), we have

$$\eta^{[y]} = \varphi_y - \xi_y u_x + (\varphi_u - \tau_y)u_y - \xi_u u_x u_y - \tau_u u_y^2 \tag{2.48}$$

which implies

$$D_y\eta^{[y]} = (\tfrac{\partial}{\partial y} + u_y\tfrac{\partial}{\partial u} + u_{xy}\tfrac{\partial}{\partial u_x} + u_{yy}\tfrac{\partial}{\partial u_y})\eta^{[y]}. \tag{2.49}$$

Substituting (2.26) in (2.49) we have

$D_y\eta^{[y]} = \phi_{yy} - \xi_{yy}u_x + (2\phi_{uy} - \tau_{yy})u_y - 2\xi_{uy}u_x u_y + (\phi_{uu} - 2\tau_{uy})u_y^2 - \xi_{uu}u_x u_y^2 - \tau_{uu}u_y^3 - \xi_y u_{xy} + (\phi_u - \tau_y)u_{yy} - \xi_u u_{yy}u_x - 2\tau_u u_{yy}u_y - \xi_u u_{xy}u_y.$

Finally,

$\eta^{[yy]} = \phi_{yy} - \xi_{yy}u_x + (2\phi_{uy} - \tau_{yy})u_y - 2\xi_{uy}u_x u_y + (\phi_{uu} - 2\tau_{uy})u_y^2 - \xi_{uu}u_x u_y^2 - \tau_{uu}u_y^3 - 2\xi_y u_{xy} + (\phi_u - 2\tau_y)u_{yy} - \xi_u u_{yy}u_x - 3\tau_u u_{yy}u_y - 2\xi_u u_{xy}u_y.$

## 2.2 Procedure of finding symmetries of Partial differential equations

Given a 2nd PDE

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{xy}) = 0, \tag{2.50}$$

with the symmetry

$$x^* = f(x, y, u; \epsilon), \qquad y^* = g(x, y, u; \epsilon), \qquad u^* = h(x, y, u; \epsilon).$$

Treating the equation (2.50) as an algebraic equation in $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{xy}$ and considering the symmetry in operator form

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u}, \tag{2.51}$$

one arrives at the invariance criterion for the symmetries of (2.50).

**Definition 2.2** *Consider the PDE (2.50), the operator (2.51) is a symmetry of (2.50) if*

$$X^{[n]} F|_{F=0} = 0. \tag{2.52}$$

*Equation (2.52) is called invariance criterion for symmetries of PDE (2.50) and is the basis for computing the symmetries of PDE (2.27)*

The invariance criterion generally leads us to an over determined system of linear Partial differential equations in $\xi(x, y, u)$, $\tau(x, y, u)$ and $\phi(x, y, u)$. This system is often known as a system of determining equations and its solution is the set of all possible infinitesimals $\xi(x, y, u)$, $\tau(x, y, u)$ and $\phi(x, y, u)$ that satisfy the invariance condition.

### 2.2.1 Steps for finding symmetries of PDEs of the form (2.50)

Utilizing the invariance criterion, the symmetries can be found through the following steps.

1. Consider a symmetry generator $X$ of the form (2.52).

2. Find the 2nd prolongation $X^{[2]}$.

3. Apply the prolongation found in step (1) to a given PDE (2.50), and substitute the constraint to obtain $X^{[2]}F|_{F=0} = 0$.

4. Compare the coefficients of the derivative of $u$ from step (3) to find the determining equations.

5. Simplify and solve the determining equations to get the corresponding symmetries i.e $\xi(x, y, u)$, $\tau(x, y, u)$ and $\phi(x, y, u)$.

As an example, we will go through steps $(1 - 4)$ to find the determining equation of Poisson equation on surfaces of revolution.

### 2.2.2 Determining equations of Poisson equation on surfaces of revolution

In this section we present the system of determining equations which will be used to complete group classification of Poisson equation on surfaces of revolution in the subsequent chapters. Consider the Poisson equation

$$f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} = g(u),$$

on surfaces of revolution. The infinitesimal symmetry transformations have the form

$$x^* = x + \epsilon\xi(x, y) + O(\epsilon),$$

$$y^* = y + \epsilon\tau(x, y) + O(\epsilon),$$

$$u^* = u + \epsilon\phi(x, y) + O(\epsilon).$$

The corresponding generator of the symmetry algebra is of the form

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}.$$

We need up to the second prolongation containing information on how and $x$, $u_x$, $u_{xx}$ and $u_{yy}$ are transformed and this is given by

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}.$$

Let

$$F(x, y, u, u_x, u_{xx}, u_{yy}) = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} - g(u).$$

Applying invariance criteria we have

$$X^{[2]}(f'(x)u_x + u_{xx} + e^{f(x)}u_{yy} - g(u))|_{u_{xx}=g(u)-e^{f(x)}u_{yy}-f'(x)u_x} = 0. \tag{2.53}$$

This implies that

$$\xi f_{xx}u_x - 2\xi f_x e^{-2f}u_{yy} - \phi g_u g_u + \eta^{[x]}f_x + \eta^{[xx]} + \eta^{[yy]}e^{-2f} = 0. \tag{2.54}$$

From the previous section, we obtained

$\eta^{[yy]} = \phi_{yy} - \xi_{yy}u_x + (2\phi_{uy} - \tau_{yy})u_y - 2\xi_{uy}u_x u_y + (\phi_{uu} - 2\tau_{uy})u_y^2 - \xi_{uu}u_x u_y^2 - \tau_{uu}u_y^3 - 2\xi_y u_{xy} + (\phi_u - 2\tau_y)u_{yy} - \xi_u u_{yy}u_x - 3\tau_u u_{yy}u_y - 2\xi_u u_{xy}u_y$

$\eta^{[x]} = \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y$ and

$\eta^{[xx]} = \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xy} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_u u_{xy}u_x.$

Substituting above into (2.54), we have

$\xi f_{xx}u_x - 2\xi f_x e^{-2f}u_{yy} - \phi g_u + (\phi_x + (\phi_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y)f_x + \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xy} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_u u_{xy}u_x + (\phi_{yy} - \xi_{yy}u_x + (2\phi_{uy} - \tau_{yy})u_y - 2\xi_{uy}u_x u_y + (\phi_{uu} - 2\tau_{uy})u_y^2 - \xi_{uu}u_x u_y^2 - \tau_{uu}u_y^3 - 2\xi_y u_{xy} + (\phi_u - 2\tau_y)u_{yy} - \xi_u u_{yy}u_x - 3\tau_u u_{yy}u_y - 2\xi_u u_{xy}u_y)e^{-2f} = 0.$

Substituting the constraint i.e $u_{xx} = -e^{-2f}u_{yy} - f_x u_x + g(u)$, we have

$\xi f_{xx}u_x - 2\xi f_x e^{-2f}u_{yy} - \phi g_u + (\phi_x + (\phi_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y)f_x + \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)(-e^{-2f}u_{yy} - f_x u_x + g(u)) - 2\tau_x u_{xy} - 3\xi_u(-e^{-2f}u_{yy} - f_x u_x + g(u))u_x - \tau_u(-e^{-2f}u_{yy} - f_x u_x + g(u))u_y - 2\tau_u u_{xy}u_x + (\phi_{yy} - \xi_{yy}u_x + (2\phi_{uy} - \tau_{yy})u_y - 2\xi_{uy}u_x u_y + (\phi_{uu} - 2\tau_{uy})u_y^2 - \xi_{uu}u_x u_y^2 - \tau_{uu}u_y^3 - 2\xi_y u_{xy} + (\phi_u - 2\tau_y)u_{yy} - \xi_u u_{yy}u_x - 3\tau_u u_{yy}u_y - 2\xi_u u_{xy}u_y)e^{-2f} = 0.$

Comparing the coefficient of the derivative of $u$, we have

$u_0$: $\phi_x f_x + \phi_{xx} + e^{-2f(x)}\phi_{yy} + g(u)(\phi_u - 2\xi_x) - g_u\phi = 0$

$u_x$: $\xi f_{xx} + 2\phi_{ux} - \xi_{xx} + \xi_x f_x - e^{-2f(x)}\xi_{yy} - 3g\xi_u = 0$

$u_y$: $e^{-2f(x)}\tau_{yy} + \tau_{xx} + f_x\tau_x - 2\phi_{uy} - g\tau_u = 0$

$u_x^2$: $-2\xi_u f_x + (\phi_{uu} - 2\xi_{xu} + 3\xi_u f_x) = 0$

$u_y^2$: $(\phi_{uu} - 2\tau_{yu})e^{-2f(x)} = 0$

$u_{xy}$: $-2e^{-2f(x)}\xi_y - 2\tau_x = 0$

$u_x u_y$: $-2\tau_{ux} - 2\xi_{uy}e^{-2f(x)} = 0$

$u_x^3$: $\xi_{uu}$

$u_y^3$: $\tau_{uu} = 0$

$u_{yy}$: $-2\xi f_x e^{-2f(x)} - e^{-2f(x)}(\phi_u - 2\xi_x) + e^{-2f(x)}(\phi_u - 2\tau_y) = 0$

$u_x^2 u_y$: $\tau_{uu} = 0$

$u_x u_y^2$: $\xi_{uu} = 0$

$u_x u_{yy}$: $-3e^{-f(x)}\xi_u + 2e^{-2f(x)}\xi_u$

$u_y u_{yy}$: $-3e^{-f(x)}\tau_u + 2e^{-2f(x)}\tau_u = 0$

$u_y u_{xy}$: $-2e^{-2f(x)}\xi_u = 0$

$u_x u_{yx}$: $-2\tau_u = 0$

Assuming a lexicographic ordering $\phi > \tau > \xi > f > g$ and $x > y > u$, we have the following eight (8) determining equations:

e1: $\xi_u = 0$

e2: $\tau_u = 0$

e3: $-\xi f_x + \xi_x - \tau_y = 0$

e4: $e^{-2f(x)}\xi_y + \tau_x = 0$

e5: $\phi_{uu} = 0$

e6: $e^{-2f(x)}\tau_{yy} + \tau_{xx} + f_x\tau_x - 2\phi_{uy} = 0$

e7: $\xi f_{xx} + 2\phi_{ux} - \xi_{xx} + \xi_x f_x - e^{-2f(x)}\xi_{yy} = 0$

e8: $\phi_x f_x + \phi_{xx} + e^{-2f(x)}\phi_{yy} + g(u)(\phi_u - 2\xi_x) - g_u\phi = 0$

We are going to use these determining equations directly in the following chapter to find the minimal symmetry algebra and to investigate the group classification problem for Poisson equation on surfaces of revolution.

## CHAPTER 3

# MINIMAL SYMMETRY ALGEBRAS OF THE POISSON EQUATION ON SURFACES OF REVOLUTION

The aim of this chapter is to determine the minimal symmetry algebras of the Poisson equation on surfaces of revolution. The calculations are based on a triangulation of the determining equations for the symmetries of the general Poisson equation on surfaces of revolution. Precisely the following result is obtained:

**Theorem 3.1** *Let $M^2$ be a surface of revolution with parametrization*

$$X(x, y) = (v(x), e^{f(x)} \cos(y), e^{f(x)} \sin(y)),$$

*The minimal symmetry algebra $\mathcal{L}$ of the non-linear Poisson equation*

$$f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} = g(u)$$

*on $M^2$, for all non-linear $g(u)$, consists of the following classes:*

*(i) If $M^2$ is a surfaces of non-constant curvature, then $\mathcal{L}$ is 1-dimensional and is generated by $X_1 = \frac{\partial}{\partial y}$.*

*(ii) For surfaces $M^2$ with constant curvature, $\mathcal{L}$ is 3-dimensional generated by $\mathcal{L} = \langle X_1, X_2, X_3 \rangle$.*

*(a) Assume $M$ is a flat surface*

- *if $f(x) = \ln|c_1|$ i.e $M^2$ is a cylinder, then*

$$X_1 = y\frac{\partial}{\partial x} - \frac{x}{c_1^2}\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y}$$

- *if $f(x) = \ln|x + c_1|$ i.e $M^2$ is a plane, then*

$$X_1 = \sin(y)\frac{\partial}{\partial x} + \frac{\cos(y)}{(c_2 + x)}\frac{\partial}{\partial y}, \qquad X_2 = \cos(y)\frac{\partial}{\partial x} - \frac{\sin(y)}{(c_2 + x)}\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial y}$$

- *if $f(x) = \ln|l(x + c_2)|$ i.e $M^2$ is a cone, then*

$$X_1 = \sin(ly)\frac{\partial}{\partial x} + \frac{\cos(ly)}{l(x + d)}\frac{\partial}{\partial y}, \qquad X_2 = \cos(ly)\frac{\partial}{\partial x} - \frac{\sin(ly)}{l(d + x)}\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial y}.$$

*(b) If $M^2$ is a surfaces of constant positive curvature with $f(x) = \ln|a\cos(bx)|$ i.e a sphere $(ab = 1)$, or a surface of spindle type $ab < 1$ or a surface of bulge type $(ab > 1)$ then*

$$X_1 = \sin(aby)\frac{\partial}{\partial x} - \tan(bx)\cos(aby)\frac{\partial}{\partial y},$$

$$X_2 = \cos(aby)\frac{\partial}{\partial x} + \tan(bx)\sin(aby)\frac{\partial}{\partial y},$$

$$X_3 = \frac{\partial}{\partial y}.$$

*(c) If $M^2$ is a surface of constant negative curvature*

- *if $f(x) = bx + c$ i.e $M^2$ is a pseudosphere, then*

$$X_1 = y\frac{\partial}{\partial x} + \frac{1}{2b}\left(\frac{e^{-2(bx+c)}}{b} - b^2y^2\right)\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x} - by\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial y}$$

- *if $f(x) = \ln|b\cosh(rx)|$ i.e $M^2$ is a hyperbolic, then*

$$X_1 = e^{rby}\frac{\partial}{\partial x} - \tanh(rx)e^{rby}\frac{\partial}{\partial y}$$

$$X_2 = e^{-rby}\frac{\partial}{\partial x} + \tanh(rx)e^{-rby}\frac{\partial}{\partial y}$$

$$X_3 = \frac{\partial}{\partial y}$$

- *if $f(x) = \ln|b\sinh(rx)|$ i.e $M^2$ is of conic type, then*

$$X_1 = \sin(rby)\frac{\partial}{\partial x} + \cos(rby)arctanh(rx)\frac{\partial}{\partial y},$$

$$X_2 = \cos(rby)\frac{\partial}{\partial x} - \sin(rby)arctanh(rx)\frac{\partial}{\partial y},$$

$$X_3 = \frac{\partial}{\partial y}.$$

The proof of the theorem 3.1 is contained in section 3.1. In section 3.2, we further analyze the determining equations and list forms of $g(u)$ for which the Poisson equation (3.1) may admit larger symmetry algebras. These forms will be investigated in chapter 4 and 5 to completely answer the group classification question for Poisson equation on surfaces of revolution.

## 3.1 Minimal symmetry algebras

Recall from chapter 1 that Poisson equation $\Delta u = g(u)$ on any surface of revolution with parametrization $X(x, y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y))$, can be written as

$$f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} = g(u). \tag{3.1}$$

In order to obtain the symmetries of Poisson equation (3.1), we take generator of the symmetry algebras of the form

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}.$$

If $X^{[2]}$ represents the second prolongation of $X$, then as derived in section 2.2.2 the invariance condition

$$X^{[2]}(f'(x)u_x + u_{xx} + e^{f(x)}u_{yy} - g(u))|_{u_{xx}=g(u)-e^{f(x)}u_{yy}-f'(x)u_x} = 0,$$

generates the following system of eight determining equations:

e1: $\xi_u = 0$

e2: $\tau_u = 0$

e3: $-\xi f_x + \xi_x - \tau_y = 0$

e4: $e^{-2f(x)}\xi_y + \tau_x = 0$

e5: $\phi_{uu} = 0$

e6: $e^{-2f(x)}\tau_{yy} + \tau_{xx} + f_x\tau_x - 2\phi_{uy} = 0$

e7: $\xi f_{xx} + 2\phi_{ux} - \xi_{xx} + \xi_x f_x - e^{-2f(x)}\xi_{yy} = 0$

e8: $\phi_x f_x + \phi_{xx} + e^{-2f(x)}\phi_{yy} + g(u)(\phi_u - 2\xi_x) - g_u\phi = 0$

Next we carry out a triangulation of the determining equation to obtain the minimal symmetry algebras. For systematic triangulation, we assume a lexicographic ordering as $\phi > \tau > \xi > f > g$.

Using $(e4)_y + (e3)_x$, we have

e9: $-\xi f_{xx} + \xi_{xx} - \xi_x f_x + e^{-2f(x)}\xi_{yy} = 0.$

Putting $e9$ in $e7$, we have

e10: $\phi_{ux} = 0$

By $(e4)_x - e^{-2f(x)}(e3)_y$ we have

e11: $e^{-2f(x)}\tau_{yy} + \tau_{xx} - e^{-2f(x)}f_x\xi_y = 0.$

Substituting in $e_6$ and using $e_4$, gives

$e_{12} : \phi_{uy} = 0.$

Differentiating $e_8$ with respect to $u$, we have

$e_{13} : 2g_u\xi_x + g_{uu}\phi = 0.$

Differentiating $e_{13}$ by $u$ again, we have

$e_{14} : 2g_{uu}\xi_x + g_{uu}\phi_u + g_{uuu}\phi = 0.$

Differentiating $e_{14}$ with respect to $u$, gives

$e_{15} : 2g_{uuu}\xi_x + 2g_{uuu}\phi_u + g_{uuuu}\phi = 0.$

Using $e_{15}$ and $e_{14}$ to eliminate $\phi_u$, we have

$e_{16} : (g_{uu}g_{uuuu} - 2g_{uuu}^2)\phi - 2g_{uu}g_{uuu}\xi_x = 0.$

Using $e_{13}$ and $e_{16}$ to eliminate $\phi$, we haves

$e_{17} : (g_u g_{uu} g_{uuuu} - 2g_{uuu}^2 g_u + g_{uu}^2 g_{uuu})\xi_x = 0.$

Differentiate $e_{13}$ with respect to $x$, gives

$e_{18} : 2g_u\xi_{xx} + g_{uu}\phi_x = 0.$

Differentiate $e_{18}$ with respect to $u$, gives

$e_{19} : 2g_{uu}\xi_{xx} + g_{uuu}\phi_x = 0.$

Using $e_{18}$ and $e_{19}$ to eliminate $\phi_x$, we have

$e_{20} : (g_{uu}^2 - g_u g_{uuu})\xi_{xx} = 0.$

To find the minimal symmetry algebra with no condition on $g(u)$, we see that $e_{17}$

implies

$$\xi_x = 0,$$

and from $e_{13}$

$$\phi = 0.$$

Rewriting the determining equations, using $\xi_x = 0$ and $\phi = 0$, we have

ee1: $\xi_u = 0$

ee2: $\tau_u = 0$

ee3: $\xi f_x + \tau_y = 0$

ee4: $e^{-2f(x)}\xi_y + \tau_x = 0$

ee5: $0 = 0$

ee6: $e^{-2f(x)}\tau_{yy} + \tau_{xx} + f_x\tau_x = 0$

ee7: $\xi f_{xx} - e^{-2f(x)}\xi_{yy} = 0$

ee8: $0 = 0$

Next, we analyze these equations to find minimal symmetry algebras that exist for all $g(u)$. Differentiate $(ee7)$ with respect to $x$ and again using $ee7$ to eliminate $\xi_{yy}$, we have

$$(f_{xxx} + 2f_x f_{xx})\xi = 0. \tag{3.2}$$

Next we determine minimal symmetry algebras according to different possibilities of $f(x)$.

**Case 1:** Surfaces $X(x,y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y))$ with $f_{xxx} + 2f_x f_{xx} \neq 0$

Since the Gaussian curvature of surfaces with unit speed parameterize by

$$X(x,y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y))$$

is defined by

$$k = -(f_{xx} + f_x^2).$$

The class of surfaces with $f_{xxx} + 2f_x f_{xx} \neq 0$, corresponds to surfaces of revolution with non-constant curvature. If $(f_{xxx} + 2f_x f_{xx}) \neq 0$, implies by (3.2)

$$\xi = 0. \tag{3.3}$$

Now solving the determining equations with $\xi = 0$, $\phi = 0$ gives the minimal algebra for Poisson equation on surfaces of revolution with non-constant curvature as

$$\xi = 0, \qquad \tau = k_1, \qquad \phi = 0.$$

**Case 2:** Surfaces $X(x,y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y))$ with $f_{xxx} + 2f_x f_{xx} = 0$

This class consists of surfaces of revolution of constant curvature, where

$$f_{xxx} + 2f_x f_{xx} = 0, \tag{3.4}$$

implies that

$$f_{xx} + f_x^2 = k. \tag{3.5}$$

For further analysis, we treat the sub-cases $k \neq 0$ and $k = 0$ separately.

**case 2.1:** $k \neq 0$

We determine the unit speed surface of revolution with parametrization

$$X(x, y) = (v(x), e^{f(x)} \cos(y), e^{f(x)} \sin(y))$$

, satisfying

$$f_{xx} + f_x^2 = k \neq 0 \tag{3.6}$$

i.e the surfaces of revolution of non-zero constant curvature, along with the corresponding minimal symmetry algebra of the Poisson equation on these surfaces.

We begin by obtaining the forms of $\xi$ and $\tau$ for surfaces with $f(x)$ satisfying equation (3.4), these forms will be utilized later to determine the minimal symmetry algebras for specific cases.

Equation (3.4) leads to the identity

$$\frac{d(e^{2f(x)} f_{xx})}{dx} = e^{2f(x)}(f_{xxx} + 2f_x f_{xx}) = 0. \tag{3.7}$$

This implies that $e^{2f(x)}f_{xx}$ is constant, say $e^{2f(x)}f_{xx} = c$ with $c \in R$. This will gives three possibilities; $f_{xx} < 0$, $f_{xx} = 0$ and $f_{xx} > 0$.

Furthermore, from $ee_7$ we have

$$\xi_{yy} - e^{2f} f_{xx}\xi = 0 \qquad or \qquad \xi_{yy} - c\xi = 0. \tag{3.8}$$

The solutions of equation (3.8) for different possibilities are as follows.

- For $f_{xx} < 0$, we have $e^{2f(x)}f_{xx} = -n^2$

$$\xi(y) = k_1 sin(ny) + k_2 cos(ny). \tag{3.9}$$

- For $f_{xx} = 0$, we have $e^{2f(x)}f_{xx} = 0$

$$\xi(y) = k_1 y + k_2. \tag{3.10}$$

- For $f_{xx} > 0$, implies $e^{2f(x)}f_{xx} = n^2$

$$\xi(y) = k_1 e^{ny} + k_2 e^{-ny}. \tag{3.11}$$

Once $\xi$ is know, $\tau$ can be found by using $e_4$. From $e_4$, we have

$$\tau(x, y) + \xi_y \int e^{-2f(x)}dx = r(y). \tag{3.12}$$

Putting (3.12) in $e_3$, gives

$$r_y = \xi_{yy}\left(\int e^{-2f(x)}dx\right) - \xi f_x,$$

which implies

$$r(y) = \int \left(\xi_{yy}\int e^{-2f(x)}dx - f_x\xi\right) dy + \gamma. \tag{3.13}$$

Substituting (3.13) in (3.12), gives

$$\tau(x,y) = \int \left(\xi_{yy}\int e^{-2f(x)}dx - f_x\xi\right) dy - \xi_y \int e^{-2f(x)}dx + \gamma. \tag{3.14}$$

Next we analyze different solutions of equation (3.6) and then determine the corresponding surfaces $X(x,y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y))$, as well as minimal symmetry algebras of Poisson equation on these surfaces using (3.9), (3.10), (3.11) and (3.14). Recall that for all case we have $\phi = 0$.

**Case 2.1.1:** $f(x)$ is linear i.e $f_{xx} = 0$ and $f_x \neq 0$. This gives $f(x) = m^{-1}x + c$ for some constant $m$ and $c$ with $0 < |m| < \infty$. Without loss of generality, let $c = \ln|m|$ then $w(x) = |m|e^{x/m}$ since $(v'(x))^2 + (w'(x))^2 = 1$, implies $v(x) = \int_0^x \sqrt{1 - e^{\frac{2t}{m}}}dt$; with $-\infty < x < 0$ if $m > 0$ and $0 \leqslant x < \infty$ if $m < 0$.

Assume $|m| = a$ then, the curve $\alpha(x)$ takes the form

$$\alpha(x) = \begin{cases} (\int_0^x \sqrt{1 - e^{\frac{2t}{a}}}dt, ae^{x/a}) & \text{if } -\infty < x < 0 \\ (\int_0^x \sqrt{1 - e^{\frac{-2t}{a}}}dt, ae^{-x/a}) & \text{if } 0 \leq x < \infty. \end{cases} \tag{3.15}$$

This correspond to unit speed parametrization of **tractrix** and its corresponding surface of revolution is a **pseudosphere** or **tractoid** as in [39] with regular parametrization,

$$X(x, y) = \begin{cases} (\int_0^x \sqrt{1 - e^{\frac{2t}{a}}}dt, ae^{x/a}\cos(y), ae^{x/a}\sin(y)) & \text{if } -\infty < x < 0,\ 0 \leq y < 2\pi \\ (\int_0^x \sqrt{1 - e^{\frac{-2t}{a}}}dt, ae^{-x/a}\cos(y), ae^{-x/a}\sin(y)) & \text{if } 0 \leq x < \infty,\ 0 \leq y < 2\pi. \end{cases} \tag{3.16}$$

The Gaussian curvature of pseudosphere in this case is negative $K = -a^{-2}$ with $f_{xx} = 0$.

For $m^{-1} = b$, i.e for pseudosphere $f(x) = bx + c$, we obtain from (3.10) and (3.14)

$$\xi = k_1 y + k_2$$

,

$$\tau = \frac{k_1}{2b}e^{-2(bx+c)} - \frac{by}{2}(k_1 y + 2k_2) + k_3$$

. Since $\phi = 0$, the minimal symmetry algebra is 3-dimensional and is generated by

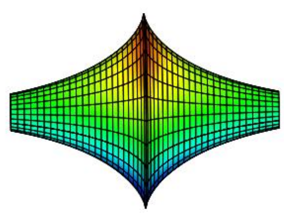
$$X_1 = y\frac{\partial}{\partial x} + \frac{1}{2b}\left(\frac{e^{-2(bx+c)}}{b} - b^2y^2\right)\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x} - by\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $bX_1$ | $-X_2$ |
| $X_2$ | $-bX_1$ | $0$ | $bX_3$ |
| $X_3$ | $X_2$ | $-bX_3$ | $0$ |

Table 3.1: Commutator table of minimal symmetries of Poisson equation on pseudosphere

Figure 3.1: Below is the illustration of pseudosphere with $a = 4$ and $-8 \leq x \leq 8$



**Case 2.1.2:** $f_{xx} + f_x^2 = k \neq 0$ with $f_{xx} \neq 0$. Substituting $f_x = F(x)$ in (3.5), we have

$$F^2 + F_x = k. \tag{3.17}$$

Now let us look for possible cases resulting from different non-zero values of $k$.

**Case 2.1.2.1:** $f_{xx} + f_x^2 = k \neq 0$ with $f_{xx} \neq 0$ and $k < 0$. Let $k = -n^2$ with $n \neq 0$, therefore

from (3.17), we have

$$F^2 + F_x = -n^2, \tag{3.18}$$

$$F = f_x = -n\tan(nx + nc). \tag{3.19}$$

Therefore from (3.19), we have

$$f(x) = \ln|b\cos(nx + nc)|. \tag{3.20}$$

Without loss of generality, we assume $c = 0$ and $n = a^{-1}$, the unit speed curve $\alpha$ is given by

$$\alpha(x) = \left(\int_0^{\frac{x}{a}} \sqrt{a^2 - b^2\sin^2 t}dt, |b\cos(a^{-1}x)|\right), \tag{3.21}$$

with $x$ having the following ranges:

If $b = a$ then $-\frac{1}{2}\pi a < x < \frac{1}{2}\pi a$.

If $b < a$ then $-\frac{1}{2}\pi a < x < \frac{1}{2}\pi a$.

If $b > a$ then $-a\arcsin\frac{a}{b} < x < a\arcsin\frac{a}{b}$.

This generate the class of surfaces of revolution $S(a, b)$ with patch

$$X(x, y) = \left(\int_0^{\frac{x}{a}} \sqrt{a^2 - b^2\sin^2 t}dt, b\cos(a^{-1}x)\cos(y), b\cos(a^{-1}x)\sin(y)\right), \quad 0 \leq y < 2\pi,$$

having a constant positive Gaussian curvature $K = a^{-2}$ with $f_{xx} < 0$. Depending on the values of $a$ and $b$ this class $S(a,b)$ consists of following type of surfaces [39].

- $S(a,a)$ is an ordinary sphere with radius $a$.
- If $0 < a < b$, $S(a,b)$ is a surface of revolution like a barrel sharp vertices and does not meet axis of revolution (Bulge type).
- If $0 < b < a$, $S(a,b)$ is a surface of revolution like a rugby ball with sharp vertices on its axis of revolution (Spindle type).

For $f(x) = \ln|a\cos(bx)|$, we have $f_{xx} = -b^2\sec^2(bx) < 0$. Hence from (3.9) and (3.14), we obtain

$$\xi(y) = k_1 \sin(aby) + k_2 \cos(aby),$$

$$\tau(x,y) = -\tan(bx)\cos(aby)k_1 + \tan(bx)\sin(aby)k_2 + k_3.$$

Since $\phi = 0$, the minimal symmetry algebra for this class is 3-dimensional and is generated by

$$X_1 = \sin(aby)\frac{\partial}{\partial x} - \tan(bx)\cos(aby)\frac{\partial}{\partial y},$$

$$X_2 = \cos(aby)\frac{\partial}{\partial x} + \tan(bx)\sin(aby)\frac{\partial}{\partial y},$$

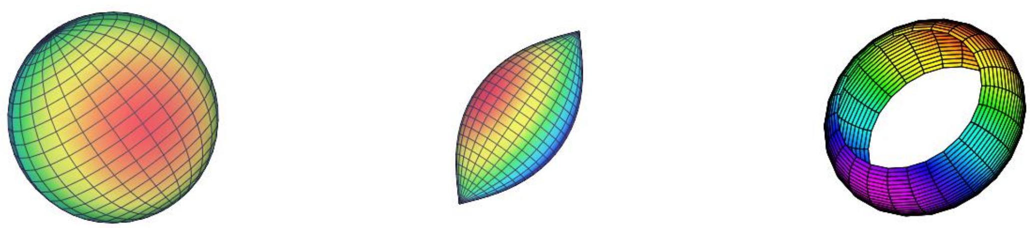
$$X_3 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $\frac{b}{a}X_3$ | $-abX_2$ |
| $X_2$ | $\frac{-b}{a}X_3$ | $0$ | $abX_1$ |
| $X_3$ | $abX_2$ | $-abX_1$ | $0$ |

Table 3.2: Commutator table of minimal symmetries of Poisson equation on class of surfaces S(a,b)

Figure 3.2: Illustration of $S(a,b)$ with $a = b = 1$, $a = 1.5$ and $b = 1$, $a = 1$ and $b = 1.5$ respectively



**Case 2.1.2.2:** $f_{xx} + f_x^2 = k \neq 0$ $f_{xx} \neq 0$ and $k > 0$. Let $k = r^2$ with $r \neq 0$. From (3.17), we have

$$F^2 + F_x = r^2. \tag{3.22}$$

Integrating (3.22), gives

$$\ln\left|\frac{F + r}{F - r}\right| = 2r(x + p). \tag{3.23}$$

We know from (3.23) either $f_{xx} > 0$ or $f_{xx} < 0$. Therefore from (3.22) we have the following two cases

**Case 2.1.2.2.1:** $f_{xx} + f_x^2 = k \neq 0$ with $f_{xx} \neq 0$, $k > 0$ and $f_{xx} > 0$. Therefore we have $F_x > 0$ and from (3.22) $F^2 - r^2 < 0$, or equivalently $(F+r)(F-r) < 0$. This implies $\frac{F+r}{F-r} < 0$ and hence from (3.23)

$$\ln\left(\frac{F+r}{r-F}\right) = 2r(x+p), \tag{3.24}$$

$$F = f_x = \frac{r(e^{2r(x+p)} - 1)}{e^{2r(x+p)} + 1} = \frac{r\sinh(r(x+p))}{\cosh(r(x+p))}, \tag{3.25}$$

which implies $f(x) = \ln(b.\cosh(r(x+p)))$ with $b > 0$. Without loss of generality, let us assume $p = 0$ and $r = a^{-1}$ $(a > 0)$ . The unit speed curve $\alpha$ take the form

$\alpha(x) = (\int_0^{\frac{x}{a}} \sqrt{a^2 - b^2 \sinh^2 t} dt, b\cosh(a^{-1}x)) \qquad -a \operatorname{arcsin} h\frac{a}{b} < x \leq a \operatorname{arcsin} h\frac{a}{b}$.

This generates the surface of revolution with regular parametrization

$$X(x,y) = \left(\int_0^{\frac{x}{a}} \sqrt{a^2 - b^2\sinh^2 t} dt, b\cosh(a^{-1}x)\cos(y), b\cosh(a^{-1}x)\sin(y)\right) \quad 0 \leq y < 2\pi.$$

These surfaces are of **hyperboloid type**[39] and have constant negative Gaussian curvature $K = -a^{-2}$ with $f_{xx} > 0$.

For $f(x) = \ln(b\cosh(rx))$, we have $f_{xx} = r^2 sech^2(rx) > 0$. Hence from (3.11) and (3.14), we obtain

$$\xi(x) = k_1 e^{rby} + k_2 e^{-rby},$$

$$\tau(x,y) = -e^{rby}\tanh(rx)k_1 + e^{rby}\tanh(rx)k_2 + k_3.$$

Since $\phi = 0$, the minimal symmetry algebra for Poisson equation on surfaces of revolution of hyperboloid type with $f(x) = \ln b \cosh(rx)$ is 3-dimensional and is generated by

$$X_1 = e^{rby}\frac{\partial}{\partial x} - \tanh(rx)e^{rby}\frac{\partial}{\partial y},$$

$$X_2 = e^{-rby}\frac{\partial}{\partial x} + \tanh(rx)e^{-rby}\frac{\partial}{\partial y},$$

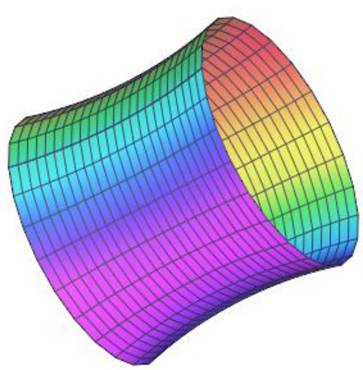
$$X_3 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $\frac{2r}{b}X_3$ | $-rbX_1$ |
| $X_2$ | $\frac{-2r}{b}X_3$ | $0$ | $rbX_2$ |
| $X_3$ | $rbX_1$ | $-rbX_2$ | $0$ |

Table 3.3: Commutator table of minimal symmetries of hyperboloid type

Figure 3.3: Illustration of a surface of revolution of hyperboloid type with $a = 6$ and $b = 3$



**Case 2.1.2.2.2:** $f_{xx} + f_x^2 = k \neq 0$ $f_{xx} \neq 0$, $k > 0$ and $f_{xx} < 0$. Therefore we have $F_x < 0$

and from (3.22) $F^2 - r^2 > 0$, or $(F + r)(F - r) > 0$. This implies $\frac{F+r}{F-r} > 0$ and hence from (3.23)

$$\ln\left(\frac{F+r}{F-r}\right) = 2r(x+p), \tag{3.26}$$

$$F = f_x = \frac{r(e^{2r(x+p)}+1)}{e^{2r(x+p)}-1} = \frac{r\cosh(r(x+p))}{\sinh(r(x+p))}, \tag{3.27}$$

which implies $f(x) = \ln|b\sinh(r(x+p))|$. Without loss of generality, let us assume $p = 0$ and $r = a^{-1}$ $(a \neq 0)$. Therefore the unit speed curve $\alpha$ take the form

$\alpha(x) = (\int_0^{\frac{x}{a}} \sqrt{a^2 - b^2\cosh^2 t}dt, b\sinh(a^{-1}x)) \qquad 0 < x \leq a\arcsin\frac{\sqrt{b^2-a^2}}{b}$.

This generates the surfaces of revolution with regular parametrization

$$X(x,y) = \left(\int_0^{\frac{x}{a}} \sqrt{a^2 - b^2\cosh^2 t}dt, b\sinh(a^{-1}x)\cos(y), b\sinh(a^{-1}x)\sin(y)\right) \quad 0 \leq y < 2\pi.$$

These surfaces are known as surfaces of revolution of **conic type**[39] and have a constant negative Gaussian curvature $K = -a^{-2}$.

For $f(x) = \ln(b\sinh(rx))$, we have $f_{xx} = -r^2 csch^2(rx) < 0$. Hence from (3.9) and (3.14), we obtain

$$\xi(y) = k_1\sin(rby) + k_2\cos(rby),$$

$$\tau(x,y) = k_1 arctanh(rx)\cos(rby) - k_2 arctanh(rx)\sin(rby) + k_3.$$

Since $\phi = 0$, the minimal symmetry algebra of Poisson equation for this case is 3-dimensional and is generated by

$$X_1 = \sin(rby)\frac{\partial}{\partial x} + \cos(rby)arctanh(rx)\frac{\partial}{\partial y},$$

$$X_2 = \cos(rby)\frac{\partial}{\partial x} - \sin(rby)arctanh(rx)\frac{\partial}{\partial y},$$

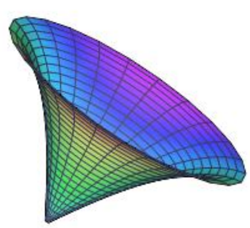
$$X_3 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $\frac{-r}{b}X_3$ | $-rbX_2$ |
| $X_2$ | $\frac{r}{b}X_3$ | $0$ | $rbX_1$ |
| $X_3$ | $rbX_2$ | $-rbX_1$ | $0$ |

Table 3.4: Commutator table of minimal symmetries of Poisson equation of conic type

Figure 3.4: Illustration of a surfaces of revolution of **conic type** with $a = 4$ and $b = 2$



**Case 2.2:** $k = 0$ i.e surfaces with $f_{xx} + f_x^2 = k = 0$

The expression of the Gaussian curvature implies that these are surfaces of zero Gaussian curvature i.e flat surface of revolution. It is well-known that a flat surface of revolution is either cylinder or plane or cone. Solving $f_{xx} + f_x^2 = 0$, we have

$$f(x) = \ln|c_1 x + c_2|.$$

This gives all the flat surfaces of revolution, the values $c_1 = 0$ and $c_1 = 1$ respectively give **cylinder** and **plane**, otherwise the generated surface of revolution is **cone**.

The minimal symmetry algebras of Poisson equation on flat surfaces of revolution are provided below.

- **Cylinder :** For $f(x) = \ln|c_1|$, we have $f_{xx} = 0$. Hence from (3.10) and (3.14), we obtain

  $$\xi(y) = k_1 y + k_2,$$

  $$\tau(x, y) = -\frac{k_1 x}{c_1^2} + k_3.$$

  Since $\phi = 0$, the minimal symmetry algebra of Poisson equation for cylinder is 3-dimensional and is generated by

  $$X_1 = y\frac{\partial}{\partial x} - \frac{x}{c_1^2}\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y}$$

  with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $\frac{1}{c_1^2}X_3$ | $-X_2$ |
| $X_2$ | $-\frac{1}{c_1^2}X_3$ | $0$ | $0$ |
| $X_3$ | $X_2$ | $0$ | $0$ |

Table 3.5: Commutator table of minimal symmetries of Poisson equation on Cylinder

- **Plane :** For $f(x) = \ln|c_2 + x|$, we have $f_{xx} = -(c_2 + x)^{-2} < 0$. Hence from (3.9) and (3.14), we obtain

$$\xi(y) = k_1 \sin(y) + k_2 \cos(y),$$

$$\tau(x, y) = k_1 \frac{\cos(y)}{(c_2 + x)} - k_2 \frac{\sin(y)}{(c_2 + x)} + k_3.$$

Since $\phi = 0$, the minimal symmetry algebra of Poisson equation for plane is 3-dimensional and is generated by

$$X_1 = \sin(y)\frac{\partial}{\partial x} + \frac{\cos(y)}{b(c_2 + x)}\frac{\partial}{\partial y}, \qquad X_2 = \cos(y)\frac{\partial}{\partial x} - \frac{\sin(y)}{(c_2 + x)}\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $0$ | $-X_2$ |
| $X_2$ | $0$ | $0$ | $X_1$ |
| $X_3$ | $X_2$ | $-X_1$ | $0$ |

Table 3.6: Commutator table of minimal symmetries of Poisson equation on plane

- **Cone** For $f(x) = \ln|l(x + d)|$, we have $f_{xx} = -(d + x)^{-2} < 0$. Hence from (3.9) and

(3.14), we obtain

$$\xi(y) = k_1 \sin(ly) + k_2 \cos(ly),$$

$$\tau(x,y) = k_1 \frac{\cos(ly)}{l(d+x)} - k_2 \frac{\sin(ly)}{l(d+x)} + k_3.$$

Since $\phi = 0$, the minimal symmetry algebra of Poisson equation for cone is 3-dimensional and is generated by with

$$X_1 = \sin(ly)\frac{\partial}{\partial x} + \frac{\cos(ly)}{l(x+d)}\frac{\partial}{\partial y}, \qquad X_2 = \cos(ly)\frac{\partial}{\partial x} - \frac{\sin(ly)}{l(d+x)}\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | 0 | 0 | $-lX_2$ |
| $X_2$ | 0 | 0 | $lX_1$ |
| $X_3$ | $lX_2$ | $-X_1$ | 0 |

Table 3.7: Commutator table of minimal symmetries of Poisson equation on cone

## 3.2 Generation of different forms of $g(u)$ for possible larger symmetry algebras

To search for non-linear functions $g(u)$ that may give larger symmetry algebra, we analyze equation $e_{17}$.

Since for any choice of $g(u)$, $\xi_x = 0$ leads to minimal algebra, we assume $\xi_x \neq 0$ and solve this equation

$$(g_u g_{uu} g_{uuuu} - 2g_{uuu}^2 g_u + g_{uu}^2 g_{uuu}) = 0. \tag{3.28}$$

For finding solution of (3.28), substitute $g_u = H$ in (3.28) to have

$$HH_u H_{uuu} - 2HH_{uu}^2 + H_u^2 H_{uu} = 0. \tag{3.29}$$

Therefore we have the following cases:

- If $H_{uu} = 0$, then $g(u)$ is quadratic i.e

$$g(u) = au^2 + bu + c. \tag{3.30}$$

- If $H_{uu} \neq 0$, divide (3.29) by $HH_u H_{uu}$, gives

$$\frac{H_{uuu}}{H_{uu}} - 2\frac{H_{uu}}{H_u} + \frac{H_u}{H} = 0, \tag{3.31}$$

$$\ln\left|\frac{H_{uu}H}{H_u^2}\right| = k. \tag{3.32}$$

Integrating (3.32), we get

$$\frac{H_{uu}}{H_u} - \frac{cH_u}{H} = 0 \tag{3.33}$$

Integration (3.33), implies $\ln|\frac{H_u}{H^d}| = k_1$, therefore $\frac{H_u}{H^d} = k_2$ and we have the following cases:

- If $d = 1$, we get $g(u)$ of the form

$$g(u) = ae^{bu} + c. \tag{3.34}$$

- If $d = 2$ , then $H$ is of the form $H = (au + b)^{-1}$, which implies

$$g(u) = \frac{\ln|au + b|}{a} + c. \tag{3.35}$$

- If $d \neq 1, 2$, then $H$ is of the form $H = (au + b)^{\frac{1}{1-c}}$ which gives

$$g(u) = (au + b)^n + c \qquad n \neq 0, 1, 2. \tag{3.36}$$

CHAPTER 4

# GROUP CLASSIFICATION OF POISSON EQUATION ON SURFACES OF REVOLUTION WITH CONSTANT GAUSSIAN CURVATURE

## 4.1 Introduction

The aim of this chapter is to study the complete group classification problem for the nonlinear Poisson equation on surfaces of revolution of constant curvature. Precisely, the following result is obtained.

**Theorem 4.1** *Let $M^2$ be a surfaces of revolution parametrized by*

$$X(x,y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y)),$$

*with constant curvature $\mathcal{K}$, i.e $f(x)$ satisfies*

$$f_{xx} + f_x^2 = k = -\mathcal{K}. \tag{4.1}$$

*The 3-dimensional minimal symmetry algebra of the non-linear Poisson equation*

$$\Delta u = g(u)$$

*on $M^2$ extends in the following cases:*

*(i)* $g(u) = (au+b)^2$

*The algebra extends to 4-dimensional for flat surfaces of revolution.*

*(a) For the cylinder with $f(x) = 0$, the algebra is generated by*

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2au+b}{a}\frac{\partial}{\partial u}, \qquad X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial y}.$$

*(b) For the plane with $f(x) = \ln|c_2 + x|$, the algebra is generated by*

$$X_1 = (x - c_2)\frac{\partial}{\partial x} - \frac{2au + b}{a}\frac{\partial}{\partial u}, \qquad X_2 = \cos(y)\frac{\partial}{\partial x} + \frac{\sin(y)}{x + c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(y)\frac{\partial}{\partial x} - \frac{\cos(y)}{x + c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}.$$

*(c) For the cone with $f(x) = \ln|l(x + d)|$, the algebra is generated by*

$$X_1 = (x - c_2)\frac{\partial}{\partial x} - \frac{2au + b}{a}\frac{\partial}{\partial u}, \qquad X_2 = \cos(ly)\frac{\partial}{\partial x} + \frac{\sin(yl)}{x + c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(ly)\frac{\partial}{\partial x} - \frac{\cos(ly)}{x + c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}.$$

*(ii) $g(u) = (au + b)^n$ $n \neq 0, 1, 2$*

*The algebra extends to 4-dimensional for flat surfaces of revolution.*

*(a) For the cylinder with $f(x) = 0$, the algebra is generated by*

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2(au + b)}{a(n - 1)}\frac{\partial}{\partial u}, \qquad X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial y}.$$

*(b) For the plane with $f(x) = \ln|c_2 + x|$, the algebra is generated by*

$$X_1 = (x + c_2)\frac{\partial}{\partial x} - \frac{2(au + b)}{a(n - 1)}\frac{\partial}{\partial u}, \qquad X_2 = \cos(y)\frac{\partial}{\partial x} + \frac{\sin(y)}{x + c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(y)\frac{\partial}{\partial x} - \frac{\cos(y)}{x + c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}.$$

*(c) For the cone with $f(x) = \ln|l(x+d)|$, the algebra is generated by*

$$X_1 = (x+c_2)\frac{\partial}{\partial x} - \frac{2(au+b)}{a(n-1)}\frac{\partial}{\partial u}, \qquad X_2 = \cos(ly)\frac{\partial}{\partial x} + \frac{\sin(yl)}{x+c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(ly)\frac{\partial}{\partial x} - \frac{\cos(ly)}{x+c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}.$$

*(iii)* $g(u) = ae^{bu} - \frac{2k}{b}$

*The symmetry algebra is infinite dimensional algebras generated by*

$$X = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} + \phi(x,y)\frac{\partial}{\partial u}.$$

*Where $\tau(x,y)$, is a harmonic function on $M^2$ satisfying*

$$e^{-2f}\tau_{yy} + \tau_{xx} + f_x\tau_x = 0.$$

*The function $\xi(x,y)$, is given by the relations*

$$-\xi f_x + \xi_x - \tau_y = 0,$$

$$e^{-2f}\xi_y + \tau_x = 0.$$

*And the function $\phi(x,y)$ is given by*

$$\phi = \frac{-2\xi_x}{b}.$$

The proof of the Theorem 4.1 is provided in section 4.2. The group classification for Poisson equation on surfaces of non-constant curvature is investigate in chapter 5.

## 4.2 Proof of group classification theorem

It was shown in section 3.1 that, the minimal symmetry algebra of non-linear Poisson equation on surfaces of revolution of constant curvature is 3-dimensional. This result was achieved via an analysis of the triangulation of the determining equations of symmetries of Poisson equation. The triangulation contained the equation

$$(g_u g_{uu} g_{uuuu} - 2g_{uuu}^2 g_u + g_{uu}^2 g_{uuu})\xi_x = 0. \tag{4.2}$$

which will play a key role in the analysis here. As shown in section 3.2, the only non-linear solutions of

$$g_u g_{uu} g_{uuuu} - 2g_{uuu}^2 g_u + g_{uu}^2 g_{uuu} = 0 \tag{4.3}$$

are:

- $$g(u) = au^2 + bu + c, \tag{4.4}$$

- $$g(u) = (au+b)^n + c \qquad\qquad n \neq 0, 1, 2, \tag{4.5}$$

- $$g(u) = \frac{\ln|au+b|}{a} + c, \tag{4.6}$$

- $$g(u) = ae^{bu} + c. \tag{4.7}$$

Recall from chapter 3 that $\xi_x = 0$ leads to minimal symmetry algebra. So to look for larger symmetry algebras, we assume $\xi_x \neq 0$ and analyze different possibilities of $g(u)$ along with (4.2).

### 4.2.1 Case 1.1 : $g(u)$ quadratic i.e $g(u) = au^2 + bu + c$

Substituting $g_u = 2au + b$ and $g_{uu} = 2a$ in $e_{13}$ and using $e_{20}$ respectively gives

$$\phi = \frac{-\xi_x(2au+b)}{a}, \tag{4.8}$$

$$\xi_{xx} = 0. \tag{4.9}$$

Differentiating (4.8) with respect to $x$ implies $\phi_x = 0$. Also differentiating (4.8) twice with respect to $y$ gives

$$\phi_{yy} = \frac{-\xi_{xyy}(2au + b)}{a}. \tag{4.10}$$

We also have from (4.8)

$$\phi_u = -2\xi_x.$$

From $e_9$, we have

$$\xi_{yy} = e^{2f}\xi_x f_x + \xi f_{xx} e^{2f}. \tag{4.11}$$

Differentiating (4.11) with respect to $x$ and putting in (4.10), gives

$$\phi_{yy} = \frac{-e^{2f}(2au + b)}{a}\big(\xi_x(2f_{xx} + 2f_x^2) + \xi(f_{xxx} + 2f_x f_{xx})\big). \tag{4.12}$$

Putting (4.12) in $e_8$, we have

$$\left(-\frac{(2au + b)}{a}(2f_{xx} + 2f_x^2) + \frac{b^2}{a} - 4c\right)\xi_x - \frac{(2au + b)}{a}\left(f_{xxx} + 2f_x f_{xx}\right)\xi = 0. \tag{4.13}$$

Since for surfaces of constant curvature, we have $f_{xxx}+2f_xf_{xx}=0$ which implies $f_{xx}+f_x^2=k$. It follows that $k=0$ gives flat surfaces of revolution and $k\neq 0$ leads to non-flat surfaces of revolution of constant curvature.

Then from (4.13)

$$\left(-\frac{(2au+b)}{a}(2k)+\frac{b^2}{a}-4c\right)\xi_x=0. \tag{4.14}$$

Differentiating (4.14) with respect to $u$, gives

$$(-2(2k))\xi_x=0. \tag{4.15}$$

Equation (4.15), implies $k=0$ i.e flat surfaces of revolution. From which using (4.14), we obtain

$$\left(\frac{b^2}{a}-4c\right)\xi_x=0. \tag{4.16}$$

Hence the larger algebra is possible for all **flat surfaces of revolution**, if $\frac{b^2}{a}-4c=0$ i.e if $g(u)$ is a perfect square. We assume $g(u)=(au+b)^2$, then from $e_{20}$

$$\xi=a_1(y)x+a_2(y). \tag{4.17}$$

From $\phi_u = -2\xi_x$ and using $e_{12}$, equation (4.17), becomes

$$\xi = k_1 x + a_2(y), \tag{4.18}$$

and then (4.8) gives

$$\phi = \frac{-k_1(2au+b)}{a}. \tag{4.19}$$

Finally, $e_3$ and $e_4$ imply

$$\tau = \int (-\xi f_x + \xi_x)dy + \int (-e^{-2f(x)}\xi_y - (\int (-\xi_x f_x - \xi f_{xx})dy))dx + \gamma. \tag{4.20}$$

Next, we explicitly determine the larger symmetry algebra for non-linear Poisson equation with $g(u) = (au+b)^2$ on cylinder, plane and cone.

(i) Consider cylinder with $f(x) = 0$

Using (4.18) and the value of $f(x)$ in $e_7$ we have

$$a_2(y) = k_2 y + k_3.$$

Therefore from (4.18), we have

$$\xi = k_1 x + k_2 y + k_3. \tag{4.21}$$

Using (4.21) and $f(x)$ in (4.20), we have

$$\tau = k_1 y - k_2 x + k_4. \tag{4.22}$$

Therefore we obtain 4-dimensional symmetry algebra generated by

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2au+b}{a}\frac{\partial}{\partial u}, \qquad X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| [X,X] | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 0 | 0 | $-X_3$ | $-X_4$ |
| $X_2$ | 0 | 0 | $X_4$ | $-X_3$ |
| $X_3$ | $X_3$ | $-X_4$ | 0 | 0 |
| $X_4$ | $X_4$ | $X_3$ | 0 | 0 |

Table 4.1: Commutator table of symmetry of Poisson equation on cylinder

(ii) Consider plane with $f(x) = ln|c_2 + x|$

Using (4.18) and the value of $f(x)$ in $e_7$ we have

$$a_2''(y) + a_2(y) = k_1 c_2. \tag{4.23}$$

The solution of (4.23) is

$$a_2(y) = k_2 \cos(y) + k_3 \sin(y) + k_1 c_2 \tag{4.24}$$

Therefore from (4.18), we have

$$\xi = k_1 x + k_2 \cos(y) + k_3 \sin(y) + k_1 c_2. \tag{4.25}$$

Using (4.25) and $f(x)$ in (4.20), we have

$$\tau = \frac{k_3 \cos(y) - k_2 \sin(y)}{c_2 + x} + k_4. \tag{4.26}$$

Therefore we obtain 4-dimensional symmetry algebra generated by

$$X_1 = (x + c_2)\frac{\partial}{\partial x} - \frac{2au + b}{a}\frac{\partial}{\partial u}, \qquad X_2 = \cos(y)\frac{\partial}{\partial x} + \frac{\sin(y)}{x + c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(y)\frac{\partial}{\partial x} - \frac{\cos(y)}{x + c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 0 | $-X_2$ | $-X_3$ | 0 |
| $X_2$ | $X_2$ | 0 | 0 | $X_3$ |
| $X_3$ | $X_3$ | 0 | 0 | $-X_2$ |
| $X_4$ | 0 | $-X_3$ | $X_2$ | 0 |

Table 4.2: Commutator table of symmetry of Poisson equation on plane

(iii) Consider Cone with $f(x) = \ln|l(d+x)|$

Using (4.18) and the value of $f(x)$ in $e_7$ we have

$$a_2''(y) + l^2 a_2(y) = l^2 k_1 c_2. \tag{4.27}$$

The solution of (4.27) is

$$a_2(y) = k_2 \cos(ly) + k_3 \sin(ly) + k_1 c_2. \tag{4.28}$$

Therefore from (4.18), we have

$$\xi = k_1 x + k_2 \cos(ly) + k_3 \sin(ly) + k_1 c_2. \tag{4.29}$$

Using (4.29) and $f(x)$ in (4.20), we have

$$\tau = \frac{-k_2 \sin(yl) + k_3 \cos(yl)}{c_2 + x} + k_4. \tag{4.30}$$

Therefore we obtain 4-dimensional symmetry algebra generated by

$$X_1 = (x + c_2)\frac{\partial}{\partial x} - \frac{2au + b}{a}\frac{\partial}{\partial u}, \qquad X_2 = \cos(ly)\frac{\partial}{\partial x} + \frac{\sin(yl)}{x + c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(ly)\frac{\partial}{\partial x} - \frac{\cos(ly)}{x + c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 0 | $-X_2$ | $-X_3$ | 0 |
| $X_2$ | $X_2$ | 0 | 0 | $X_3$ |
| $X_3$ | $X_3$ | 0 | 0 | $-X_2$ |
| $X_4$ | 0 | $-X_3$ | $X_2$ | 0 |

Table 4.3: Commutator table of symmetry of Poisson equation on Cone

### 4.2.2 Case 1.2: $g(u) = (au+b)^n + c$ for $n \neq 0, 1, 2$

Substituting $g_u = an(au+b)^{n-1}$ and $g_{uu} = a^2n(n-1)(au+b)^{n-2}$ in $e_{13}$ and $e_{20}$ respectively, gives

$$\phi = \frac{-2\xi_x(au+b)}{a(n-1)} \tag{4.31}$$

$$\xi_{xx} = 0. \tag{4.32}$$

Differentiating (4.31) with respect to $x$, implies $\phi_x = 0$. Also differentiating (4.31) twice with respect to $y$, gives

$$\phi_{yy} = \frac{-2\xi_{xyy}(au+b)}{a(n-1)}. \tag{4.33}$$

We also have from (4.31)

$$\phi_u = \frac{-2\xi_x}{(n-1)}.$$

And from $e_9$, we have

$$\xi_{yy} = e^{2f}\xi_x f_x + \xi f_{xx} e^{2f}. \tag{4.34}$$

Differentiating (4.34) with respect to $x$ and putting in (4.33), gives

$$\phi_{yy} = \frac{-2e^{2f}(au+b)}{a(n-1)}(\xi_x\left(2f_{xx}+2f_x^2\right) + \xi(f_{xxx}+2f_x f_{xx})). \tag{4.35}$$

Putting (4.35) in $e_8$, we have

$$\left(-2\frac{(au+b)}{a}(2f_{xx}+2f_x^2) - 2nc\right)\xi_x - 2\frac{(au+b)}{a}\left(f_{xxx}+2f_x f_{xx}\right)\xi = 0. \tag{4.36}$$

Since for surfaces of constant curvature, we have $f_{xxx}+2f_x f_{xx} = 0$ which implies $f_{xx}+f_x^2 = k$. It follows that $k = 0$ gives flat surfaces of revolution and $k \neq 0$ leads to non-flat surfaces of revolution of constant curvature.

Therefore from (4.36), we have

$$\left(-2\frac{(au+b)}{a}(2k) - 2nc\right)\xi_x = 0. \tag{4.37}$$

Differentiate (4.37) with respect to $u$, gives

$$(-2(2k))\xi_x = 0. \tag{4.38}$$

Equation (4.38), implies $k = 0$ i.e flat surfaces of revolution. From which using (4.37), we obtain

$$(-2nc)\xi_x = 0. \tag{4.39}$$

Hence the larger algebra is possible for all **flat surfaces of revolution**, if $c = 0$ i.e if $g(u) = (au + b)^n \qquad n \neq 0, 1, 2$

Therefore using $e_2 0$, $e_{12}$ and the fact that $\phi_u = \frac{-2\xi_x}{(n-1)}$, gives

$$\xi = k_1 x + a_2(y). \tag{4.40}$$

And then using (4.31), gives

$$\phi = \frac{-2k_1(au + b)}{a(n-1)}. \tag{4.41}$$

Finally, $e_3$ and $e_4$ imply

$$\tau = \int (-\xi f_x + \xi_x)\, dy + \int (-e^{-2f(x)}\xi_y - (\int (-\xi_x f_x - \xi f_{xx}) dy)) dx + \gamma. \tag{4.42}$$

Next, we explicitly determine the larger symmetry algebra for non-linear Poisson equation with $g(u) = (au + b)^n \qquad n \neq 0, 1, 2$ on cylinder, plane and cone.

(i) Consider Cylinder with $f(x) = 0$

Using (4.40) and the value of $f(x)$ in $e_7$, we have

$$a_2(y) = k_2 y + k_3.$$

Therefore from (4.40)

$$\xi = k_1 x + k_2 y + k_3. \tag{4.43}$$

Using (4.43) and $f(x)$ in (4.42), we have

$$\tau = k_1 y - k_2 x + k_4. \tag{4.44}$$

Therefore, we obtain 4-dimensional symmetry algebra generated by

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2(au+b)}{a(n-1)}\frac{\partial}{\partial u}, \qquad X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial y}.$$

(ii) Consider Plane with $f(x) = ln|c_2 + x|$

Using (4.40) and the value of $f(x)$ in $e7$, we have

$$a_2''(y) + a_2(y) = k_1 c_2. \tag{4.45}$$

The solution of (4.45) is

$$a_2(y) = k_2\cos(y) + k_3\sin(y) + k_1c_2. \tag{4.46}$$

Therefore (4.40), implies

$$\xi = k_1x + k_2\cos(y) + k_3\sin(y) + k_1c_2. \tag{4.47}$$

Using (4.47) and $f(x)$ in (4.42), we have

$$\tau = \frac{k_3\cos(y) - k_2\sin(y)}{c_2 + x} + k_4. \tag{4.48}$$

Therefore, we obtain 4-dimensional symmetry algebra generated by

$$X_1 = (x + c_2)\frac{\partial}{\partial x} - \frac{2(au + b)}{a(n-1)}\frac{\partial}{\partial u}, \qquad X_2 = \cos(y)\frac{\partial}{\partial x} + \frac{\sin(y)}{x + c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(y)\frac{\partial}{\partial x} - \frac{\cos(y)}{x + c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}.$$

(iii) Consider Cone with $f(x) = ln|l(d + x)|$

Using (4.40) and the value of $f(x)$ in $e_7$, we have

$$a_2''(y) + l^2a_2(y) = l^2k_1c_2. \tag{4.49}$$

The solution of (4.49) is

$$a_2(y) = k_2\cos(ly) + k_3\sin(ly) + k_1c_2. \tag{4.50}$$

Therefore from (4.40), we have

$$\xi = k_1x + k_2\cos(ly) + k_3\sin(ly) + k_1c_2. \tag{4.51}$$

Using (4.51) and $f(x)$ in (4.42), we have

$$\tau = \frac{-k_2\sin(yl) + k_3\cos(yl)}{c_2 + x} + k_4. \tag{4.52}$$

Therefore, we obtain 4-dimensional symmetry algebra generated by

$$X_1 = (x + c_2)\frac{\partial}{\partial x} - \frac{2(au+b)}{a(n-1)}\frac{\partial}{\partial u}, \qquad X_2 = \cos(ly)\frac{\partial}{\partial x} + \frac{\sin(yl)}{x+c_2}\frac{\partial}{\partial y}$$
$$X_3 = \sin(ly)\frac{\partial}{\partial x} - \frac{\cos(ly)}{x+c_2}\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}.$$

### 4.2.3 Case 1.3 : $g(u) = \frac{\ln|au+b|}{a} + c$

Substituting $g_u = \frac{1}{(au+b)}$ and $g_{uu} = \frac{-a}{(au+b)^2}$ in $e_{13}$ and $e_{20}$ respectively, gives

$$\phi = \frac{2\xi_x(au+b)}{a}, \tag{4.53}$$

$$\xi_{xx} = 0. \tag{4.54}$$

Differentiating (4.53) with respect to $x$ implies $\phi_x = 0$. Also differentiating (4.53) twice with respect to $y$, gives

$$\phi_{yy} = \frac{2\xi_{xyy}(au+b)}{a}. \tag{4.55}$$

We also have from (4.53)

$$\phi_u = 2\xi_x.$$

From $e_9$, we have

$$\xi_{yy} = e^{2f}\xi_x f_x + \xi f_{xx} e^{2f}. \tag{4.56}$$

Differentiating (4.56), with respect to $x$ and putting in (4.55), gives

$$\phi_{yy} = \frac{2e^{2f}(au+b)}{a}\big(\xi_x(2f_{xx} + 2f_x^2) + \xi(f_{xxx} + 2f_x f_{xx})\big). \tag{4.57}$$

Putting (4.57) in $e_8$, gives

$$((2f_{xx} + 2f_x^2)(au+b) - 1)\xi_x + ((f_{xxx} + 2f_x f_{xx})(au+b))\xi = 0. \tag{4.58}$$

Since for surfaces of revolution with constant curvature we have $f_{xxx} + 2f_x f_{xx} = 0$ which implies $f_{xx} + f_x^2 = k$. Then from (4.58)

$$((au + b)(2k) - 1)\xi_x = 0 \tag{4.59}$$

This is a contradiction, then $f_{xx} + f_x^2 \neq k$

### 4.2.4 Case 1.4 : $g(u) = ae^{bu} + c$, with $a, b \neq 0$

In this case $\xi_{xx}$ is not necessarily zero. Substituting $g_u = abe^{bu}$ and $g_{uu} = ab^2 e^{bu}$ in $e_{13}$, gives

$$\phi = \frac{-2\xi_x}{b}. \tag{4.60}$$

Differentiating (4.60) with respect to $x$, implies $\phi_x = \frac{-2\xi_{xx}}{b}$. Also differentiating (4.60) twice with respect to $y$, gives

$$\phi_{yy} = \frac{-2\xi_{xyy}}{b}. \tag{4.61}$$

From equation (4.60) and $e_1$, we get

$$\phi_u = 0.$$

From $e_9$, we have

$$\xi_{yy} = e^{2f}\xi_x f_x + \xi f_{xx} e^{2f} - \xi_{xx} e^{2f}. \tag{4.62}$$

Differentiating (4.62) with respect to $x$ and putting in (4.61), gives

$$\phi_{yy} = \frac{-2(\xi_{xx}f_x + \xi_x f_{xx} + 2f_x^2\xi_x + f_{xx}\xi_x + \xi f_{xxx} + 2f_x\xi f_{xx} - \xi_{xxx} - 2f_x\xi_{xx})}{b}. \tag{4.63}$$

Putting (4.63) in $e_8$, we have

$$((2f_{xx} + 2f_x^2 + cb))\xi_x + (f_{xxx} + 2f_x f_{xx})\xi = 0. \tag{4.64}$$

Since for surfaces of revolution with constant curvature we have $f_{xxx} + 2f_x f_{xx} = 0$, which implies $f_{xx} + f_x^2 = k$.

It follow from equation (4.64)

$$(2k + cb)\xi_x = 0 \tag{4.65}$$

Finally, from (4.65) larger symmetry algebra is possible for surfaces of revolution of constant, curvature if $bc = -2k$ i.e $g(u) = ae^{bu} - \frac{2k}{b}$ $a, b \neq 0$.

For $g(u) = ae^{bu} - \frac{2k}{b}$, it follows from determining equations that the symmetries algebra is infinite dimensional whose generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} + \phi(x,y)\frac{\partial}{\partial u},$$

and can be determined by the following procedure.

Using $e_6$, $\tau(x,y)$ is a harmonic function on the surface of revolution satisfying

$$e^{-2f}\tau_{yy} + \tau_{xx} + f_x\tau_x = 0. \tag{4.66}$$

Having known $\tau(x,y)$, $\xi = \xi(x,y)$ can then be found using $e_3$ and $e_4$ i.e

$$-\xi f_x + \xi_x - \tau_y = 0, \tag{4.67}$$

$$e^{-2f(x)}\xi_y + \tau_x = 0. \tag{4.68}$$

Finally, $\phi(x,y)$ can be found using (4.60) as

$$\phi = \frac{-2\xi_x}{b}.$$

Therefore, every harmonic function $\tau$ on surfaces of revolution of constant curvature leads to a symmetry of the Poisson equation

$$\Delta(u) = ae^{bu} - \frac{2k}{b}. \tag{4.69}$$

Below we provide some examples of symmetries of (4.69) generated by different solution of (4.66).

1. $\tau(x,y) = Constant = k_1$

Equation (4.66) is clearly satisfied. Substituting in (4.67) and (4.68), we have

$$-\xi f_x + \xi_x = 0, \tag{4.70}$$

$$\xi_y = 0. \tag{4.71}$$

From (4.71), we have $\xi = \xi(x)$. Rewriting (4.70), gives

$$\frac{\xi_x}{\xi} = f_x. \tag{4.72}$$

Integrating (4.72), gives

$$\ln|\xi| = f(x) + k_1. \tag{4.73}$$

Therefore

$$\xi = k_1 e^{f(x)}. \tag{4.74}$$

Finally, using (4.60)

$$\phi = \frac{-2k_1 f_x e^{f(x)}}{b}. \tag{4.75}$$

**Example 4.2** *Pseudosphere with $f(x) = x + c$ implies $f_{xx} + f_x^2 = 1$, therefore $k = 1$.*

*From (4.74) and (4.75) respectively we have*

$$\xi = k_1 e^x, \qquad \tau = k_1, \qquad \phi = \frac{-2k_1 x e^x}{b}.$$

**Example 4.3** *Sphere with $f(x) = \ln|\cos(x)|$ implies $f_{xx} + f_x^2 = -1$, therefore $k = -1$.*

*From (4.74) and (4.75) respectively we have*

$$\xi = k_2|\cos(x)|, \qquad \tau = k_1, \qquad \phi = \frac{2k_2 \sin(x)}{b}.$$

2. $\tau(x, y) = \tau(x)$

   Substituting in (4.66), we have

$$\tau_{xx} + \tau_x f_x = 0. \tag{4.76}$$

Solving (4.76), gives

$$\tau(x) = k_1 \int e^{-f(x)} dx + k_2. \tag{4.77}$$

Substituting (4.77) in (4.68) and (4.67), gives

$$\xi = -k_1 y e^{f(x)} + k_3 e^{f(x)}. \tag{4.78}$$

Finally, from (4.60)

$$\phi = \frac{2f_x(k_1 y e^{f(x)} - k_3 e^{f(x)})}{b}. \tag{4.79}$$

**Example 4.4** *Pseudosphere with $f(x) = x + c$ implies $f_{xx} + f_x^2 = 1$, therefore $k = 1$. From (4.77), (4.78) and (4.79) respectively, we have*

$$\xi = k_1 y e^{-x} + k_3 e^{-x} \qquad \tau = k_2 + k_1 e^{-x} \qquad \phi = \frac{2yk_1 e^{-x} + 2k_3 e^{-x}}{b}.$$

**Example 4.5** *Sphere with $f(x) = \ln|\cos(x)|$ implies $f_{xx} + f_x^2 = -1$, therefore $k = -1$. From (4.77), (4.78) and (4.79) respectively, we have*

$$\xi = -k_1 y|\cos(x)| + k_3|\cos(x)| \qquad \tau = k_1 \ln|\sec(x) + \tan(x)| + k_2$$
$$\phi = \frac{-2\tan(x)(k_1 y|\cos(x)| - k_3|\cos(x)|}{b}.$$

3. $\tau(x, y) = \tau(y)$

Substituting in (4.66), we have

$$\tau_{yy} = 0. \tag{4.80}$$

Solving (4.80), we have

$$\tau = k_1 y + k_2. \tag{4.81}$$

Equations (4.67) and (4.68), gives

$$\xi = e^{-f(x)}(k_1 \int e^{f(x)} dx + k_3). \tag{4.82}$$

Finally using (4.60)

$$\phi = -2\frac{k_1(1 + f_x e^{-f(x)}(\int e^{-f(x)} dx)) + k_3 f_x e^{-f(x)}}{b}. \tag{4.83}$$

**Example 4.6** *Pseudosphere with $f(x) = x + c$, implies $f_{xx} + f_x^2 = 1$, therefore $k = 1$. From (4.82), (4.80) and (4.83) respectively, we have*

$$\xi = -k_1 + k_3 e^x \qquad \tau = k_1 y + k_2 \qquad \phi = \frac{-2k_3 e^x}{b}.$$

**Example 4.7** *Sphere with $f(x) = \ln|\cos(x)|$ implies $f_{xx} + f_x^2 = -1$, therefore $k = -1$. From (4.82), (4.80) and (4.83) respectively, we have*

$$\xi = k_1|\cos(x)|\ln|\sec(x) + \tan(x)| + k_3|\cos(x)| \qquad \tau = k_1 y + k_2$$
$$\phi = \frac{2k_1 \sin(x)\ln|\sec(x) + \tan(x)| - k_1 + k_3 \sin(x)}{b}.$$

CHAPTER 5

# GROUP CLASSIFICATION OF POISSON EQUATION ON SURFACES OF REVOLUTION WITH NON-CONSTANT GAUSSIAN CURVATURE

## 5.1 Introduction

The aim of this chapter is to study the complete group classification problem for the non-linear Poisson equation on surfaces of revolution of non-constant curvature. Precisely the following result is obtained:

**Theorem 5.1** *Let $M^2$ be a surfaces of revolution parametrized by*

$$X(x,y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y)),$$

*with non-constant curvature i.e $f(x)$ satisfies*

$$f_{xx} + f_x^2 \neq k. \tag{5.1}$$

*The 1-dimensional minimal symmetry algebra of the non-linear Poisson equation*

$$\Delta u = g(u)$$

*on $M^2$ extended to 3-dimensional for $g(u) = ae^{bu}$ on the following surfaces*

*(a) For the helicoid, with $f(x) = \ln(\sqrt{x^2+d^2})$, the algebra is generated by*

$$X_1 = \tfrac{(x^2+d^2)}{d} sin(y)\tfrac{\partial}{\partial x} - \tfrac{xcos(y)}{d}\tfrac{\partial}{\partial y} - \tfrac{4xsin(y)}{db}\tfrac{\partial}{\partial u},$$

$$X_2 = \tfrac{(x^2+d^2)}{d} cos(y)\tfrac{\partial}{\partial x} - \tfrac{xsin(y)}{d}\tfrac{\partial}{\partial y} - \tfrac{4xcos(y)}{db}\tfrac{\partial}{\partial u},$$

$$X_3 = \tfrac{\partial}{\partial y}.$$

*(b) For the Gabriel horn, with* $f(x) = \ln|\frac{1}{x}|$*, the algebra is generated by*

$$
\begin{aligned}
X_1 &= \sqrt{2}(\tfrac{xy}{2}\tfrac{\partial}{\partial x} + (\tfrac{y^2}{2} - \tfrac{x^4}{8})\tfrac{\partial}{\partial y} - \tfrac{y}{b}\tfrac{\partial}{\partial u}),\\
X_2 &= \sqrt{2}(\tfrac{(x}{2}\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} - \tfrac{1}{b}\tfrac{\partial}{\partial u}),\\
X_3 &= \tfrac{\partial}{\partial y}.
\end{aligned}
$$

*(c) For the paraboloid, with* $f(x) = \ln(\sqrt{x})$*, the algebra is generated by*

$$
\begin{aligned}
X_1 &= -2xy\tfrac{\partial}{\partial x} + (\tfrac{x^4}{2} - 2y^2)\tfrac{\partial}{\partial y} + \tfrac{4y}{b}\tfrac{\partial}{\partial u},\\
X_2 &= -2x\tfrac{\partial}{\partial x} - 4y\tfrac{\partial}{\partial y} + \tfrac{4}{b}\tfrac{\partial}{\partial u},\\
X_3 &= \tfrac{\partial}{\partial y}.
\end{aligned}
$$

The proof of the Theorem 5.1 is provided in section 5.2.

## 5.2 Proof of group classification theorem

It was shown in section 3.1 that the minimal symmetry algebra of non-linear Poisson equation on surfaces of revolution of non-constant curvature is 1-dimensional. This result was achieved via an analysis of the triangulation of the determining equations of symmetries of Poisson equation. The triangulation contained the equation

$$(g_u g_{uu} g_{uuuu} - 2g_{uuu}^2 g_u + g_{uu}^2 g_{uuu})\xi_x = 0. \tag{5.2}$$

Which will play a key role in the analysis here.

As shown in section 3.2, that

$$g_u g_{uu} g_{uuuu} - 2g_{uuu}^2 g_u + g_{uu}^2 g_{uuu} = 0, \tag{5.3}$$

has the following four types of non-linear solutions:

- $$g(u) = au^2 + bu + c. \tag{5.4}$$

- $$g(u) = (au + b)^n + c \qquad n \neq 0, 1, 2. \tag{5.5}$$

- $$g(u) = \frac{\ln|au + b|}{a} + c. \tag{5.6}$$

- $$g(u) = ae^{bu} + c. \tag{5.7}$$

Recall from chapter 3 that $\xi_x = 0$ leads to minimal symmetry algebra. To look for larger symmetry algebras we assume $\xi_x \neq 0$ and analyze below different possibilities of $g(u)$ along with (5.2).

### 5.2.1 Case 1.1 : $g(u)$ quadratic i.e $g(u) = au^2 + bu + c$

Since for surfaces of non-constant curvature we have $f_{xxx} + 2f_x f_{xx} \neq 0$, implies $f_{xx} + f_x^2 \neq k$. Differentiating (4.13) with respect to $u$ we have

$$\frac{b^2}{a} - 4c = 0. \tag{5.8}$$

Therefore $g(u)$ has to be perfect square i.e $g(u) = (au + b)^2$, which implies from (4.13)

$$\xi_x + \frac{1}{2}\frac{(f_{xxx} + 2f_x f_{xx})}{(f_{xx} + f_x^2)}\xi = 0. \tag{5.9}$$

Solving (5.9) using the integrating factor $\mu = \sqrt{f_{xx} + f_x^2}$, (5.9) can be reduced to

$$\frac{\partial}{\partial x}\left(\xi\sqrt{f_{xx} + f_x^2}\right) = 0. \tag{5.10}$$

From (5.10), we have

$$\xi = \frac{P(y)}{\sqrt{f_{xx} + f_x^2}}. \tag{5.11}$$

Differentiating (5.11) twice with respect to $x$ and using $e_{20}$, we have

$$\frac{P(y)}{(f_{xx}+f_x^2)^{\frac{3}{2}}}\left(\frac{3}{4}\frac{(\frac{d}{dx}(f_{xx}+f_x^2))^2}{f_{xx}+f_x^2}-\frac{1}{2}\frac{d^2}{dx^2}(f_{xx}+f_x^2)\right)=0. \tag{5.12}$$

Equation (5.12) implies

$$P(y)\left(3(\frac{d}{dx}(f_{xx}+f_x^2))^2-2\frac{d^2}{dx^2}(f_{xx}+f_x^2)(f_{xx}+f_x^2)\right)=0. \tag{5.13}$$

Let $q(x)=(f_{xx}+f_x^2)$, then (5.13) becomes

$$P(y)(3{q'}^2-2q''q)=0 \tag{5.14}$$

Therefore

$$3{q'}^2-2q''q=0. \tag{5.15}$$

Solving (5.15), gives

$$\frac{3{q'}^2}{qq'}-\frac{2q''q}{qq'}=0, \tag{5.16}$$

$$\ln\left|\frac{q^3}{q'^2}\right|=c, \tag{5.17}$$

$$q^3 = \beta q'^2 \quad \beta \neq 0. \tag{5.18}$$

The solution of (5.18) is

$$q(x) = f_{xx} + f_x^2 = \frac{\gamma}{(x+B)^2} \quad \gamma \neq 0. \tag{5.19}$$

Putting (5.19) in (5.11), gives

$$\xi = \frac{P(y)(x+B)}{\gamma}. \tag{5.20}$$

Therefore from (4.8), we get

$$\phi = \frac{-P(y)(2u + \frac{b}{a})}{\gamma}. \tag{5.21}$$

Using $e_{12}$ and (5.21), we have

$$\phi_{uy} = P'(y) = 0, \tag{5.22}$$

which implies

$$P(y) = \alpha. \tag{5.23}$$

From (5.20) and (5.21) respectively, we have

$$\xi = \frac{\alpha(x+B)}{\gamma}, \tag{5.24}$$

$$\phi = \frac{-\alpha(2u+\frac{b}{a})}{\gamma}. \tag{5.25}$$

Using (5.24) in $e_4$, we have $\tau_x = 0$, which implies

$$\tau = \tau(y). \tag{5.26}$$

From $e_6$, we have

$$\tau_{yy} = 0. \tag{5.27}$$

Solving (5.27), we have

$$\tau = a_1 y + a_2. \tag{5.28}$$

Putting (5.28) in $e_3$, we get

$$\xi f_x - \xi_x + a_1 = 0. \tag{5.29}$$

Finally, putting values of $\xi$ and $f(x)$ in (5.29) to find $a_1$ and $\alpha$.. Solving for $f(x)$ from (5.19) by letting $B = 0$, we have

$$f(x) = \frac{1}{2}(\ln|x| - \sqrt{17}\ln|x| + \ln(\frac{1}{17}(x^{\sqrt{17}}c_1 - c_2)^2)). \tag{5.30}$$

Differentiating (5.30) with respect to $x$, gives

$$f_x = \frac{1}{2}\frac{c_1(1+\sqrt{17})x^{\sqrt{17}} + c_2(-1+\sqrt{17})}{(x^{\sqrt{17}}c_1 - c_2)x}. \tag{5.31}$$

Substituting (5.31) and (5.24) in (5.29), gives

$$\alpha(c_1(1+\sqrt{17})x^{\sqrt{17}} + c_2(-1+\sqrt{17})) - (4x^{\sqrt{17}}c_1 - 4c_2)(\frac{\alpha}{2} + a_1) = 0. \tag{5.32}$$

Simplifying (5.32), gives

$$c_2\left(\alpha + \alpha\sqrt{17} - 4\,a_1\right) + x^{\sqrt{17}}c_1\left(-\alpha + \alpha\sqrt{17} + 4\,a1\right) = 0. \tag{5.33}$$

Comparing the coefficient of (5.33)

$$c_2\left(\alpha + \alpha\sqrt{17} - 4\,a_1\right) = 0, \tag{5.34}$$

$$c_1\left(-\alpha + \alpha\sqrt{17} + 4a_1\right) = 0. \tag{5.35}$$

Solving (5.34) and (5.35) for $a$ and $\alpha$ simultaneously, we have $\alpha = 0$ and $a_1 = 0$. Which implies from (5.23) $P(y) = 0$ and lead to minimal algebra i.e $\xi = 0$.

### 5.2.2 Case 1.2: $g(u) = (au + b)^n + c$ for $n \neq 0, 1, 2$

Since for surfaces of non-constant curvature we have $f_{xxx} + 2f_x f_{xx} \neq 0$, which implies $f_{xx} + f_x^2 \neq k$. Differentiating (4.36) with respect to $u$ implies $c = 0$ i.e $g(u) = (au + b)^n$, with $n \neq 0, 1, 2$. From (4.36), we get

$$\xi_x + \frac{1}{2}\frac{(f_{xxx} + 2f_x f_{xx})}{(f_{xx} + f_x^2)}\xi = 0. \tag{5.36}$$

Solving (5.36) using the integrating factor $\mu = \sqrt{f_{xx} + f_x^2}$, (5.36) can reduced to

$$\frac{\partial}{\partial x}\left(\xi\sqrt{f_{xx} + f_x^2}\right) = 0. \tag{5.37}$$

From (5.37), we have

$$\xi = \frac{P(y)}{\sqrt{f_{xx} + f_x^2}}. \tag{5.38}$$

Differentiating (5.38) twice with respect to $x$ and using $e_{20}$, we have

$$\frac{P(y)}{(f_{xx} + f_x^2)^{\frac{3}{2}}}\left(\frac{3}{4}\frac{(\frac{d}{dx}(f_{xx} + f_x^2))^2}{f_{xx} + f_x^2} - \frac{1}{2}\frac{d^2}{dx^2}(f_{xx} + f_x^2)\right) = 0. \tag{5.39}$$

Equation (5.39) implies

$$P(y)\left(3(\frac{d}{dx}(f_{xx}+f_x^2))^2-2\frac{d^2}{dx^2}(f_{xx}+f_x^2)(f_{xx}+f_x^2)\right)=0 \tag{5.40}$$

Let $q(x)=(f_{xx}+f_x^2)$, then (5.40) becomes

$$P(y)(3q'^2-2q''q)=0. \tag{5.41}$$

Therefore

$$3q'^2-2q''q=0. \tag{5.42}$$

Solving (5.42), gives

$$\frac{3q'^2}{qq'}-\frac{2q''q}{qq'}=0, \tag{5.43}$$

$$\ln\left|\frac{q^3}{q'^2}\right|=c, \tag{5.44}$$

$$q^3=\beta q'^2 \quad \beta\neq 0. \tag{5.45}$$

Solution of (5.45) is

$$q(x) = f_{xx} + f_x^2 = \frac{\gamma}{(x+B)^2} \quad \gamma \neq 0 \tag{5.46}$$

Putting (5.46) in (5.38), gives

$$\xi = \frac{P(y)(x+B)}{\gamma}. \tag{5.47}$$

Therefore from (4.8), gives

$$\phi = \frac{-P(y)(2u + \frac{b}{a})}{\gamma}. \tag{5.48}$$

Using $e_{12}$ and (5.48), we have

$$\phi_{uy} = P'(y) = 0, \tag{5.49}$$

which implies

$$P(y) = \alpha. \tag{5.50}$$

From equation (5.47) and (5.48) respectively, we have

$$\xi = \frac{\alpha(x+B)}{\gamma}, \tag{5.51}$$

$$\phi = \frac{-\alpha(2u + \frac{b}{a})}{\gamma}. \tag{5.52}$$

Also using (5.51) in $e_4$, we have $\tau_x = 0$, which implies

$$\tau = \tau(y). \tag{5.53}$$

From $e_6$

$$\tau_{yy} = 0. \tag{5.54}$$

Solving (5.54), we have

$$\tau = a_1 y + a_2. \tag{5.55}$$

Putting (5.55) in $e_3$, we have

$$\xi f_x - \xi_x + a_1 = 0. \tag{5.56}$$

Finally, putting values of $\xi$ and $f(x)$ in (5.56) to find $a_1$ and $\alpha$. Solving for $f(x)$ from (5.46) by letting $B = 0$ we have

$$f(x) = \frac{1}{2}(\ln|x| - \sqrt{17}\ln|x| + \ln(\frac{1}{17}(x^{\sqrt{17}}c_1 - c_2)^2)). \tag{5.57}$$

Differentiating (5.57) with respect to $x$, gives

$$f_x = \frac{1}{2}\frac{c_1(1+\sqrt{17})x^{\sqrt{17}} + c_2(-1+\sqrt{17})}{(x^{\sqrt{17}}c_1 - c_2)x}. \tag{5.58}$$

Substituting (5.58) and (5.51) in (5.56), gives

$$\alpha(c_1(1+\sqrt{17})x^{\sqrt{17}} + c_2(-1+\sqrt{17})) - (4x^{\sqrt{17}}c_1 - 4c_2)(\frac{\alpha}{2} + a_1) = 0. \tag{5.59}$$

Simplifying (5.59), gives

$$c_2\left(\alpha + \alpha\sqrt{17} - 4\,a_1\right) + x^{\sqrt{17}}c_1\left(-\alpha + \alpha\sqrt{17} + 4\,a1\right) = 0. \tag{5.60}$$

Comparing the coefficient of $x$ from (5.60), we have

$$c_2\left(\alpha + \alpha\sqrt{17} - 4\,a_1\right) = 0, \tag{5.61}$$

$$c_1\left(-\alpha + \alpha\sqrt{17} + 4\,a_1\right) = 0. \tag{5.62}$$

Solving (5.61) and (5.62) for $a$ and $\alpha$ simultaneously, we have $\alpha = 0$ and $a_1 = 0$. Which implies from (5.50) $P(y) = 0$ and lead to minimal algebra.

### 5.2.3 Case 1.3 : $g(u) = \frac{\ln|au+b|}{a} + c$

Since for surfaces of non-constant curvature, we have $f_{xxx} + 2f_x f_{xx} \neq 0$, which implies $f_{xx} + f_x^2 \neq k$. Differentiating (4.58) with respect to $u$ gives

$$\xi_x + \frac{1}{2}\frac{(f_{xxx} + 2f_x f_{xx})}{(f_{xx} + f_x^2)}\xi = 0. \tag{5.63}$$

Solving (5.63 using the integrating factor $\mu = \sqrt{f_{xx} + f_x^2}$, equation (5.63) can be reduced to

$$\frac{\partial}{\partial x}\left(\xi\sqrt{f_{xx} + f_x^2}\right) = 0. \tag{5.64}$$

From (5.37), we have

$$\xi = \frac{P(y)}{\sqrt{f_{xx} + f_x^2}} \tag{5.65}$$

Differentiating (5.65) twice with respect to $x$ and using $e_{20}$, we have

$$\frac{P(y)}{(f_{xx} + f_x^2)^{\frac{3}{2}}}\left(\frac{3}{4}\frac{(\frac{d}{dx}(f_{xx} + f_x^2))^2}{f_{xx} + f_x^2} - \frac{1}{2}\frac{d^2}{dx^2}(f_{xx} + f_x^2)\right) = 0. \tag{5.66}$$

Equation (5.66), implies

$$P(y)\left(3(\frac{d}{dx}(f_{xx} + f_x^2))^2 - 2\frac{d^2}{dx^2}(f_{xx} + f_x^2)(f_{xx} + f_x^2)\right) = 0. \tag{5.67}$$

Let $q(x) = (f_{xx} + f_x^2)$, then (5.67) becomes

$$P(y)(3q'^2 - 2q''q) = 0. \tag{5.68}$$

Therefore

$$3q'^2 - 2q''q = 0 \tag{5.69}$$

Solving (5.69), we have

$$\frac{3q'^2}{qq'} - \frac{2q''q}{qq'} = 0, \tag{5.70}$$

$$\ln\left|\frac{q^3}{q'^2}\right| = c, \tag{5.71}$$

$$q^3 = \beta q'^2 \quad \beta \neq 0. \tag{5.72}$$

Solution of (5.72) is

$$q(x) = f_{xx} + f_x^2 = \frac{\gamma}{(x+B)^2} \quad \gamma \neq 0. \tag{5.73}$$

Putting (5.73) in (5.65), gives

$$\xi = \frac{P(y)(x+B)}{\gamma}. \tag{5.74}$$

Therefore from (4.8), we get

$$\phi = \frac{-P(y)(2u+\frac{b}{a})}{\gamma}. \tag{5.75}$$

Using $e_{12}$ and (5.75), we get

$$\phi_{uy} = P'(y) = 0, \tag{5.76}$$

which implies

$$P(y) = \alpha. \tag{5.77}$$

From equation (5.74) and (5.75) respectively, we have

$$\xi = \frac{\alpha(x+B)}{\gamma}, \tag{5.78}$$

$$\phi = \frac{-\alpha(2u+\frac{b}{a})}{\gamma}. \tag{5.79}$$

Also using (5.78) in $e_4$, we have $\tau_x = 0$, which implies

$$\tau = \tau(y). \tag{5.80}$$

From $e_6$

$$\tau_{yy} = 0. \tag{5.81}$$

Solving (5.81), we have

$$\tau = a_1 y + a_2. \tag{5.82}$$

Putting (5.82) in $e_3$, we have

$$\xi f_x - \xi_x + a_1 = 0. \tag{5.83}$$

Finally, putting values of $\xi$ and $f(x)$ in (5.83) to find $a_1$ and $\alpha$. Solving for $f(x)$ from (5.73) by letting $B = 0$, we have

$$f(x) = \frac{1}{2}(\ln|x| - \sqrt{17}\ln|x| + \ln(\frac{1}{17}(x^{\sqrt{17}}c_1 - c_2)^2)). \tag{5.84}$$

Differentiating (5.84) with respect to $x$, gives

$$f_x = \frac{1}{2}\frac{c_1(1+\sqrt{17})x^{\sqrt{17}} + c_2(-1+\sqrt{17})}{(x^{\sqrt{17}}c_1 - c_2)x}. \tag{5.85}$$

Substituting (5.85) and (5.78) in (5.83), gives

$$\alpha(c_1(1+\sqrt{17})x^{\sqrt{17}} + c_2(-1+\sqrt{17})) - (4x^{\sqrt{17}}c_1 - 4c_2)(\frac{\alpha}{2} + a_1) = 0. \tag{5.86}$$

Simplifying (5.86), gives

$$c_2\left(\alpha + \alpha\sqrt{17} - 4\,a_1\right) + x^{\sqrt{17}}c_1\left(-\alpha + \alpha\sqrt{17} + 4\,a1\right) = 0. \tag{5.87}$$

Comparing the coefficient of $x$ from (5.87) we have

$$c_2\left(\alpha + \alpha\sqrt{17} - 4\,a_1\right) = 0, \tag{5.88}$$

$$c_1\left(-\alpha + \alpha\sqrt{17} + 4\,a_1\right) = 0. \tag{5.89}$$

Solving (5.88) and (5.89) for $a$ and $\alpha$ simultaneously, we have $\alpha = 0$ and $a_1 = 0$, which implies from (5.77) $P(y) = 0$ and lead to minimal algebra.

### 5.2.4 Case 1.4 : $g(u) = ae^{bu} + c$, with $a, b \neq 0$

In this case $\xi_{xx}$ is not necessarily zero. Therefore from (4.64), we have

$$(2f_{xx} + 2f_x^2 + cb)\xi_x + (f_{xxx} + 2f_x f_{xx})\xi = 0. \tag{5.90}$$

Solving (5.90), using the integrating factor

$$\mu = \sqrt{f_{xx} + f_x^2 + \frac{cb}{2}}. \tag{5.91}$$

Therefore, equation (5.90) can be reduced to

$$\frac{\partial}{\partial x}\left(\xi\sqrt{f_{xx} + f_x^2 + \frac{cb}{2}}\right) = 0, \tag{5.92}$$

which implies

$$\xi = \frac{P(y)}{\sqrt{f_{xx} + f_x^2 + \frac{cb}{2}}}. \tag{5.93}$$

To have a larger algebra we must have $P(y) \neq 0$. Substituting values of $\xi$, $\xi_{xx}$ and $\xi_{yy}$ in $e_9$, gives

$$\frac{P''(y)}{P(y)} = e^{2f}\left(\frac{q'' - f_x q'}{(4q + 2bc)} + f_{xx} - \frac{12{q'}^2}{(4q + 2bc)^2}\right) = constant = K. \tag{5.94}$$

With $q(x) = f_{xx} + f_x^2$ and assuming $c = 0$. The solution of (5.94) for different possibilities are:

- If $K = 0$ we get $P(y)$ as

$$P(y) = k_1 y + k_2. \tag{5.95}$$

Substituting (5.95) in (5.93), gives

$$\xi = \frac{k_1 y + k_2}{\sqrt{q(x)}}. \tag{5.96}$$

Putting (5.96) in (4.60) gives

$$\phi = \frac{q'(k_1 y + k_2)}{b q^{\frac{3}{2}}}.$$

From $e_3$, we get

$$\tau = \int (-\xi f_x + \xi_x)\, dy + r_1(x), \tag{5.97}$$

substituting (5.97) in $e_4$, we have

$$r_1(x) = \int \left( -e^{-2f(x)} \xi_y - \int (-\xi_x f_x - \xi f_{xx} + \xi_{xx}) dy \right) dx + k_3. \tag{5.98}$$

Finally putting (5.98) in (5.97), we have

$$\tau = \int(-\xi f_x + \xi_x)dy + \int\left(-e^{-2f(x)}\xi_y - \int(-\xi_x f_x - \xi f_{xx} + \xi_{xx})dy\right)dx + k_3. \quad (5.99)$$

- If $K = -m^2$, we get

$$P(y) = k_1\text{sin}(my) + k_2\text{cos}(my). \quad (5.100)$$

Substituting (5.100) in (5.93), we have

$$\xi = \frac{k_1\text{sin}(my) + k_2\text{cos}(my)}{\sqrt{q}}. \quad (5.101)$$

Putting (5.101) in (4.60), gives

$$\phi = \frac{q'(k_1\text{sin}(my) + k_2\text{cos}(my))}{bq^{\frac{3}{2}}}.$$

Finally, using e3 and e4, we obtain

$$\tau = \int(-\xi f_x + \xi_x)dy + \int\left(-e^{-2f(x)}\xi_y - \int(-\xi_x f_x - \xi f_{xx} + \xi_{xx})dy\right)dx + k_3.$$

- If $K = m^2$, we get

$$P(y) = k_1 e^{my} + k_2 e^{-my}. \quad (5.102)$$

Substituting (5.102) in (5.93), we have

$$\xi = \frac{k_1 e^{my} + k_2 e^{-my}}{\sqrt{q}}. \tag{5.103}$$

Putting (5.103) in (4.60), gives

$$\phi = \frac{q'(x)(k_1 e^{my} + k_2 e^{-my})}{bq^{\frac{3}{2}}}.$$

Finally, using e3 and e4, we obtain

$$\tau = \int (-\xi f_x + \xi_x) dy + \int \left( -e^{-2f(x)} \xi_y - \int (-\xi_x f_x - \xi f_{xx} + \xi_{xx}) dy \right) dx + k_3.$$

(i) Consider helicoid $f(x) = ln\sqrt{x^2 + d^2}$ Substitute $f(x)$ in (5.94), we have

$$c^2 b^8 + 2\, c^2 x^2 b^6 + 20\, c x^2 b^3 - 2\, c^2 x^6 b^2 - 4\, b^2 + 20\, b c x^4 - c^2 x^8 - K\left(2\, b + c x^4 + 2\, c x^2 b^2 + c b^4\right)^2 = 0. \tag{5.104}$$

Conferring the coefficient of $x$ from (5.104), we have

$$x^8 : (-c^2 - Kc^2) = 0, \tag{5.105}$$

$$x^6 : (-2b^2c^2 - 4Kb^2c^2) = 0, \tag{5.106}$$

$$x^4 : (-4Kbc + 20bc - 6Kc^2b^4) = 0, \tag{5.107}$$

$$x^2 : (2c^2b^2 - 8Kcb^3 + 20cb^3 - 4Kc^2b^6) = 0, \tag{5.108}$$

$$x^0 : (c^2b^8 - Kc^2b^8 - 4Kc^2b^8 - 4Kb^2 - 4b^2 - 4Kcb^5) = 0. \tag{5.109}$$

Solving (5.105) through (5.109), gives

$$K = -1, \quad c = 0 \tag{5.110}$$

Using the fact that $K = -m^2$ we have

$$\xi(x,y) = \frac{(k_1 \sin(my) + k_2 \cos(my))(x^2 + d^2)}{d}, \tag{5.111}$$

$$\tau(x,y) = \frac{k_3 d - xmk_1 \cos(my) + xmk_2 \sin(my)}{d}, \tag{5.112}$$

$$\phi(x,y) = -4\,\frac{x(k_1 \sin(my) + k_2 \cos(my))}{db}. \tag{5.113}$$

This generate a 3-dimensional Lie symmetry algebra given by

$$\mathrm{X}_1 = \tfrac{(x^2+d^2)}{d}sin(y)\tfrac{\partial}{\partial x} - \tfrac{xcos(y)}{d}\tfrac{\partial}{\partial y} - \tfrac{4xsin(y)}{db}\tfrac{\partial}{\partial u},$$

$$\mathrm{X}_2 = \tfrac{(x^2+d^2)}{d}cos(y)\tfrac{\partial}{\partial x} - \tfrac{xsin(y)}{d}\tfrac{\partial}{\partial y} - \tfrac{4xcos(y)}{db}\tfrac{\partial}{\partial u},$$

$$\mathrm{X}_3 = \tfrac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | 0 | $X_3$ | $-X_2$ |
| $X_2$ | $-X_3$ | 0 | $X_1$ |
| $X_3$ | $X_2$ | $-X_1$ | 0 |

Table 5.1: Commutator table of symmetry of Poisson equation on helicoid

(ii) Consider Gabriel horn $f(x) = ln|\frac{1}{x}|$, substitute $f(x)$ in (5.94), we have

$$16bc + b^2c^2x^2 - 16Kx^2 - 8Kbcx^4 - Kc^2b^2x^6 = 0. \tag{5.114}$$

Conferring the coefficient of $x$ from (5.114), we have

$$x^6 : (-Kc^2b^2) = 0, \tag{5.115}$$

$$x^4 : (-8Kb) = 0, \tag{5.116}$$

$$x^2 : (b^2c^2 - 16K) = 0, \tag{5.117}$$

$$x^0 : 16bc = 0. \tag{5.118}$$

Solving (5.114) through (5.118), and using the fact $b \neq 0$ gives

$$K = 0, \quad c = 0 \tag{5.119}$$

Using the fact that $K = 0$, we have

$$\xi(x, y) = \frac{1}{2}(k_1 y + k_2)\sqrt{2}x, \tag{5.120}$$

$$\tau(x, y) = \frac{1}{2}\sqrt{2}k_1 y^2 + \sqrt{2}k_2 y - \frac{1}{8}\sqrt{2}x^4 k_1 + k_3, \tag{5.121}$$

$$\phi(x, y) = -\frac{(k_1 y + k_2)\sqrt{2}}{b}. \tag{5.122}$$

This generate a three-dimensional Lie symmetry algebra given by

$$\mathrm{X}_1 = \sqrt{2}(\tfrac{xy}{2}\tfrac{\partial}{\partial x} + (\tfrac{y^2}{2} - \tfrac{x^4}{8})\tfrac{\partial}{\partial y} - \tfrac{y}{b}\tfrac{\partial}{\partial u}),$$

$$\mathrm{X}_2 = \sqrt{2}(\tfrac{x}{2}\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} - \tfrac{1}{b}\tfrac{\partial}{\partial u}),$$

$$\mathrm{X}_3 = \tfrac{\partial}{\partial y}$$

with corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $0$ | $-\sqrt{2}X_1$ | $-X_2$ |
| $X_2$ | $\sqrt{2}X_1$ | $0$ | $-\sqrt{2}X_3$ |
| $X_3$ | $X_2$ | $\sqrt{2}X_3$ | $0$ |

Table 5.2: Commutator table of symmetry of Poisson equation Gabriel horn

(iii) Consider paraboloid $f(x) = ln\sqrt{x}$, substitute $f(x)$ in (5.94), we have

$$xbc\left(5 + 2\,x^2bc\right) + K\left(-1 + 2\,x^2bc\right)^2 = 0. \qquad (5.123)$$

Conferring the coefficient of $x$ from (5.123), we have

$$x^4 : 4\,Kb^2c^2 = 0, \qquad (5.124)$$

$$x^3 : 2\,b^2c^2 = 0, \qquad (5.125)$$

$$x^2 : -4Kbc = 0, \tag{5.126}$$

$$x^0 : K = 0. \tag{5.127}$$

Solving (5.123) through (5.127), and using the fact $b \neq 0$, gives

$$K = 0, \quad c = 0 \tag{5.128}$$

Using the fact that $K = 0$, we have

$$\xi(x, y) = -2(k_1 y + k_2)x, \tag{5.129}$$

$$\tau(x, y) = -2k_1 y^2 - 4k_2 y + \frac{1}{2}k_1 x^4 + k_3, \tag{5.130}$$

$$\phi(x, y) = \frac{4(k_1 y + k_2)}{b}. \tag{5.131}$$

This generate a three-dimensional Lie symmetry algebra given by

$$\mathrm{X}_1 = -2xy\frac{\partial}{\partial x} + (\frac{x^4}{2} - 2y^2)\frac{\partial}{\partial y} + \frac{4y}{b}\frac{\partial}{\partial u},$$

$$\mathrm{X}_2 = -2x\frac{\partial}{\partial x} - 4y\frac{\partial}{\partial y} + \frac{4}{b}\frac{\partial}{\partial u},$$

$$\mathrm{X}_3 = \frac{\partial}{\partial y}$$

With corresponding commutator table given by

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | 0 | $4X_1$ | $-X_2$ |
| $X_2$ | $-4X_1$ | 0 | $4X_3$ |
| $X_3$ | $X_2$ | $-4X_3$ | 0 |

Table 5.3: Commutator table of symmetry of Poisson equation on paraboloid

(iv) Consider torus $f(x) = ln|1 + cos(x)|$, substitute $f(x)$ in (5.94) and using $cos(x) = Y$, we have by conferring the coefficient of $Y$

$$Y^3 : -2bc + b^2c^2 = 0, \tag{5.132}$$

$$x^2 : -7\,bc + 3\,b^2c^2 + 3 - 4Kbc + Kb^2c^2 + 4K = 0, \tag{5.133}$$

$$x^1 : 6 - 4Kbc - 8\,bc + 3\,b^2c^2 + 2\,Kb^2c^2 = 0, \tag{5.134}$$

$$x^0 : 3 + b^2c^2 - 3\,bc + Kb^2c^2 = 0. \tag{5.135}$$

Looking (5.132) through (5.135), using the fact $b \neq 0$ there is no, $c$ and $K$ that will satisfying those equations, therefore from (5.93) we conclude $P(y) = 0$, which implies

$$\xi = 0 \tag{5.136}$$

and lead to minimal algebra.

## 5.3 Conclusions

We have successfully carried out group classification of non-linear Poisson equation on surfaces of revolution. It was shown in section 3.1 that the minimal symmetry algebra of non-linear Poisson equation on surfaces of revolution of non-constant curvature is 1-dimensional, and 3-dimensional for the case of surfaces non-constant curvature. This result was achieved via an analysis of the triangulation of the determining equations of symmetries of non-linear Poisson equation for all arbitrary function $g(u)$.

The symmetry algebra of non-linear Poisson equation extends for the cases given in the table below:

| $g(u)$ | Description of the surface | Number of extra symmetries |
|---|---|---|
| $g(u) = (au + b)^2$ | Flat surfaces | 4 |
| $g(u) = (au + b)^n \quad n \neq 0, 1, 2$ | Flat surfaces | 4 |
| $g(u) = ae^{bu} - \frac{2k}{b}$ | Surfaces of constant curvature | Infinite dimension |
| $g(u) = ae^{bu}$ | Paraboloid, Helicoid, Gabriel horn. | 3 |

Table 5.4: Determining functions for which extra symmetry algebras exist for the non-linear Poisson equation on surfaces of revolution.

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